

PBW-BASIS FOR UNIVERSAL ENVELOPING ALGEBRAS OF DIFFERENTIAL GRADED POISSON ALGEBRAS

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ABSTRACT. For any differential graded (DG for short) Poisson algebra A given by generators and relations, we give a “formula” for computing the universal enveloping algebra A^e of A . Moreover, we prove that A^e has a Poincaré-Birkhoff-Witt basis provided that A is a graded commutative polynomial algebra. As an application of the PBW-basis, we show that a DG symplectic ideal of a DG Poisson algebra A is the annihilator of a simple DG Poisson A -module, where A is the DG Poisson homomorphic image of a DG Poisson algebra R whose underlying algebra structure is a graded commutative polynomial algebra.

1. INTRODUCTION

The notion of Poisson algebras arises naturally in the study of Hamiltonian mechanics and Poisson geometry. Recently, many important generalizations on Poisson algebras have been obtained in both commutative and noncommutative settings: Poisson orders [1], Poisson PI algebras [13], graded Poisson algebras [4], double Poisson algebras [18], Novikov-Poisson algebras [21], Quiver Poisson algebras [23], noncommutative Leibniz-Poisson algebras [2], left-right noncommutative Poisson algebras [3], noncommutative Poisson algebras [20] and differential graded Poisson algebras [12], etc. An interesting and practical idea to develop Poisson algebras is to study Poisson universal enveloping algebras, which was first introduced by Oh in 1999 [14], and later Oh, Park and Shin studied the Poincaré-Birkhoff-Witt basis (PBW-basis for short) for Poisson universal enveloping algebras [16]. Since then, Poisson universal enveloping algebras have been studied in a series of papers [17, 22]. In particular, the second author and the third author of the present paper studied the universal enveloping algebras of Poisson Hopf algebras and Poisson Ore-extensions [10, 11]. Our main aim of this paper is to study the PBW-basis for universal enveloping algebras of DG Poisson algebras.

In [5, 6, 7, 19], Hodges, Levasseur, Joseph and Vanciliff proved that symplectic leaves of certain Poisson varieties correspond bijectively to primitive ideals of the respective quantum algebras. After that, Oh defined the symplectic ideal of a Poisson algebra and proved that there is a one to one correspondence between the primitive ideals of quantum 2×2 matrix algebra and the symplectic ideals of a Poisson algebra constructed appropriately [15]. Moreover, he and two other authors showed that the symplectic ideal of a Poisson homomorphic image of a Poisson polynomial algebra $\mathbb{K}[x_1, \dots, x_n]$ is the annihilator of a simple Poisson module. Motivated by the DG version for symplectic ideals, there is a natural question: Is a DG symplectic ideal of a DG Poisson algebra A the annihilator of a simple DG Poisson A -module? The present paper gives a positive answer.

The paper is organized as follows. In Section 2, we briefly review some basic concepts related to DG Poisson algebras, DG Poisson modules and universal enveloping algebras of DG Poisson algebras. In

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particular, we construct the universal enveloping algebra of any DG Poisson algebra given by generators and relations. Section 3 is devoted to the study of PBW-basis for universal enveloping algebras. To be specific, for a DG Poisson algebra R whose underlying algebra structure is a graded commutative polynomial algebra, by using Gröbner-Shirshov basis theory developed in [8] and [9], we prove that the universal enveloping algebra R^e has a PBW-basis, which is analogous to the PBW-basis for universal enveloping algebras of Lie algebras. In the last section, we focus on the simple DG Poisson module. As an application of the PBW-basis theorem for universal enveloping algebras of DG Poisson algebras, we prove that a DG symplectic ideal of a DG Poisson algebra A is the annihilator of a simple DG Poisson A -module, where A is the DG Poisson homomorphic image of R .

Throughout the whole paper, \mathbb{Z} denotes the set of integers, \mathbb{k} denotes a base field and everything is over \mathbb{k} unless otherwise stated, all (graded) algebras are assumed to have an identity and all (graded) modules are assumed to be unitary.

2. UNIVERSAL ENVELOPING ALGEBRAS OF DIFFERENTIAL GRADED POISSON ALGEBRAS

In this section, first we briefly review some basic definitions and properties of DG Poisson algebras and universal enveloping algebras, then we construct the universal enveloping algebra of any DG Poisson algebra A given by generators and relations.

2.1. DG Poisson algebras. By a graded algebra we mean a \mathbb{Z} -graded algebra. A DG algebra is a graded algebra with a \mathbb{k} -linear homogeneous map $d : A \rightarrow A$ of degree 1, which is also a graded derivation. Any graded algebra can be viewed as a DG algebra with differential $d = 0$; in this case it is called a DG algebra with trivial differential. Let A, B be two DG algebras and $f : A \rightarrow B$ be a graded algebra map of degree zero. Then f is called a DG algebra map if f commutes with the differentials.

Definition 2.1. Let A be a graded \mathbb{k} -vector space. If there is a \mathbb{k} -linear map

$$\{\cdot, \cdot\} : A \otimes A \rightarrow A$$

of degree 0 such that:

- (i) (graded antisymmetry): $\{a, b\} = -(-1)^{|a||b|}\{b, a\}$;
- (ii) (graded Jacobi identity): $\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{|a||b|}\{b, \{a, c\}\}$,

for any homogeneous elements $a, b, c \in A$, then $(A, \{\cdot, \cdot\})$ is called a graded Lie algebra.

Definition 2.2. [12] Let (A, \cdot) be a graded \mathbb{k} -algebra. If there is a \mathbb{k} -linear map

$$\{\cdot, \cdot\} : A \otimes A \rightarrow A$$

of degree 0 such that

- (i) $(A, \{\cdot, \cdot\})$ is a graded Lie algebra;
- (ii) (graded commutativity): $a \cdot b = (-1)^{|a||b|}b \cdot a$;
- (iii) (biderivation property): $\{a, b \cdot c\} = \{a, b\} \cdot c + (-1)^{|a||b|}b \cdot \{a, c\}$,

for any homogeneous elements $a, b, c \in A$, then A is called a graded Poisson algebra. If in addition, there is a \mathbb{k} -linear homogeneous map $d : A \rightarrow A$ of degree 1 such that $d^2 = 0$ and

- (iv) (graded Leibniz rule for bracket): $d(\{a, b\}) = \{d(a), b\} + (-1)^{|a|}\{a, d(b)\}$;
- (v) (graded Leibniz rule for product): $d(a \cdot b) = d(a) \cdot b + (-1)^{|a|}a \cdot d(b)$,

for any homogeneous elements $a, b \in A$, then A is called a DG Poisson algebra, which is usually denoted by $(A, \cdot, \{\cdot, \cdot\}, d)$, or simply by $(A, \{\cdot, \cdot\}, d)$ or A if no confusions arise.

Remark 2.3. For a DG algebra B , assume throughout the paper that B_P is the DG Poisson algebra B with the “standard graded Lie bracket”: $[a, b] = ab - (-1)^{|a||b|}ba$ for any homogeneous elements $a, b \in B$.

Note that an ideal I of DG Poisson algebra A is called a DG Poisson ideal if $d(I) \subseteq I$, $\{I, A\} \subseteq I$. Let B be another DG Poisson algebra. A graded algebra map $\rho : A \rightarrow B$ is said to be a DG Poisson algebra map if $\rho \circ d_A = d_B \circ \rho$ and $\rho(\{a, b\}_A) = \{\rho(a), \rho(b)\}_B$ for all homogeneous elements $a, b \in A$. For example, if I is a DG Poisson ideal of A , then the canonical projection $\pi_I : A \rightarrow A/I$ is a DG Poisson algebra map.

We denote by $\mathbf{DG(P)A}$ the category of DG (Poisson) algebras whose morphism space consists of DG (Poisson) algebras map.

Now, we recall the definition of DG Poisson modules over DG Poisson algebras.

Definition 2.4. [12] Let $A = (A, \cdot, \{ \cdot, \cdot \}, d) \in \mathbf{DGPA}$, and M be a left graded module over A . We call M a left DG Poisson module over A provided that

- (i) (M, ∂) is a left DG module over the DG algebra A . That is to say, there is a \mathbb{K} -linear map $\partial : M \rightarrow M$ of degree 1 such that $\partial^2 = 0$ and

$$\partial(am) = d(a)m + (-1)^{|a|}a\partial(m)$$

for all homogeneous elements $a \in A$ and $m \in M$. Here ∂ is also called the differential of M .

- (ii) M is a left graded Poisson module over the graded Poisson algebra A . That is to say, there is a \mathbb{K} -linear map $\{ \cdot, \cdot \}_M : A \otimes M \rightarrow M$ of degree 0 such that
 - (ia) $\{a, bm\}_M = \{a, b\}_A m + (-1)^{|a||b|}b\{a, m\}_M$;
 - (ib) $\{ab, m\}_M = a\{b, m\}_M + (-1)^{|a||b|}b\{a, m\}_M$, and
 - (ic) $\{a, \{b, m\}_M\}_M = \{\{a, b\}_A, m\}_M + (-1)^{|a||b|}\{b, \{a, m\}_M\}_M$,
 for all homogeneous elements $a, b \in A$ and $m \in M$.
- (iii) the \mathbb{K} -linear map ∂ is compatible with the bracket $\{ \cdot, \cdot \}_M$. That is, we have

$$\partial(\{a, m\}_M) = \{d(a), m\}_M + (-1)^{|a|}\{a, \partial(m)\}_M,$$

for all homogeneous elements $a \in A$ and $m \in M$.

Similarly, a left DG Poisson module M over a DG Poisson algebra A is usually denoted by $(M, \{ \cdot, \cdot \}_M, \partial)$, or simply by M if there are no confusions.

Definition 2.5. Let $A = (A, \cdot, \{ \cdot, \cdot \}, d) \in \mathbf{DGPA}$, and $(M, \{ \cdot, \cdot \}_M, \partial)$ be a left DG Poisson module over A . A left graded submodule $N \leq M$ is called a left DG Poisson submodule provided that $\partial(N) \subseteq N$ and $\{A, N\}_M \subseteq N$, which is usually denoted by $N \leq_p M$.

2.2. Universal enveloping algebras of DG Poisson algebras. In this subsection, we recall the definition and some properties of the universal enveloping algebra of a DG Poisson algebra $A = (A, \cdot, \{ \cdot, \cdot \}, d)$.

Definition 2.6. [12] Let $A = (A, \cdot, \{ \cdot, \cdot \}, d) \in \mathbf{DGPA}$ and $(A^{ue}, \partial) \in \mathbf{DGA}$. We call (A^{ue}, ∂) is a universal enveloping algebra of A if there exist a DG algebra map $\alpha : (A, d) \rightarrow (A^{ue}, \partial)$ and a DG Lie algebra map $\beta : (A, \{ \cdot, \cdot \}, d) \rightarrow (A^{ue}, [\cdot, \cdot], \partial)$ satisfying

$$\begin{aligned} \alpha(\{a, b\}) &= \beta(a)\alpha(b) - (-1)^{|a||b|}\alpha(b)\beta(a), \\ \beta(ab) &= \alpha(a)\beta(b) + (-1)^{|a||b|}\alpha(b)\beta(a), \end{aligned}$$

for any homogeneous elements $a, b \in A$, such that for any $(D, \delta) \in \mathbf{DGA}$ with a DG algebra map $f : (A, d) \rightarrow (D, \delta)$ and a DG Lie algebra map $g : (A, \{ \cdot, \cdot \}, d) \rightarrow (D, [\cdot, \cdot], \delta)$ satisfying

$$\begin{aligned} f(\{a, b\}) &= g(a)f(b) - (-1)^{|a||b|}f(b)g(a), \\ g(ab) &= f(a)g(b) + (-1)^{|a||b|}f(b)g(a), \end{aligned}$$

for all $a, b \in A$, then there exists a unique DG algebra map $\phi : (A^{ue}, \partial) \rightarrow (D, \delta)$, making the diagram

$$\begin{array}{ccc} (A, \{\cdot, \cdot\}, d) & \xrightarrow{\alpha, \beta} & (A^{ue}, \partial) \\ & \searrow f, g \quad \swarrow \exists! \phi & \\ & (D, \delta) & \end{array}$$

“bi-commute”, i.e., $\phi\alpha = f$ and $\phi\beta = g$.

For an $A \in \mathbf{DGPA}$, we denote by (A^{ue}, α, β) the universal enveloping algebra of A . Note that universal enveloping algebra A^{ue} of a DG Poisson algebra A was defined in order that a \mathbb{k} -vector space M is a DG Poisson A -module if and only if M is a DG A^{ue} -module and that universal enveloping algebra of A is unique up to isomorphic (see [12]).

Proposition 2.7. Let (A^{ue}, α, β) be the universal enveloping algebra of a finitely generated DG Poisson algebra A . Then α is injective.

Proof. For every $a \in A$, define $\gamma(a), \delta(a) \in \text{End}_{\mathbb{k}}(A)$ by

$$\gamma(a)(b) = ab, \delta(a)(b) = \{a, b\},$$

for all $b \in A$. As we all know, $\text{End}_{\mathbb{k}}(A)$ is a graded endomorphism ring if A is a finitely generated \mathbb{k} -module, because

$$\text{End}_{\mathbb{k}}(A) = H(A, A) = \bigoplus_{n \in \mathbb{Z}} H(A, A)_n$$

and

$$H(A, A)_n = \{\psi \in \text{Hom}_{\mathbb{k}}(A, A) \mid \psi(A_i) \subseteq A_{i+n}\}.$$

Let

$$d' \in \text{End}(\text{End}_{\mathbb{k}}(A)), d'(f)(a) = d(f(a)) - (-1)^{|f|} f(d(a)),$$

for any elements $f \in \text{End}_{\mathbb{k}}(A)$ and $a \in A$, then $\text{End}_{\mathbb{k}}(A)$ is a DG algebra. It is easy to proof that γ is a graded algebra map. Moreover, we have

$$(\gamma d)(a)(b) = \gamma(d(a))(b) = d(a)b$$

and

$$(d' \gamma)(a)(b) = d'(\gamma(a))(b) = d(\gamma(a)(b)) - (-1)^{|\gamma(a)|} \gamma(a)(d(b)) = d(ab) - (-1)^{|a|} ad(b),$$

for all $a, b \in A$, then γ is a DG algebra map by using graded Leibniz rule for product. In fact, for any homogeneous elements $a, b, c \in A$, we have

$$\begin{aligned} [\delta(a), \delta(b)](c) &= \delta(a)\delta(b)(c) - (-1)^{|a||b|} \delta(b)\delta(a)(c) \\ &= \{a, \{b, c\}\} - (-1)^{|a||b|} \{b, \{a, c\}\}, \\ \delta(\{a, b\})(c) &= \{\{a, b\}, c\}, \\ (\delta d)(a)(b) &= \delta(d(a))(b) = \{d(a), b\} \end{aligned}$$

and

$$(d' \delta)(a)(b) = d'(\delta(a))(b) = d(\delta(a)(b)) - (-1)^{|a|} \delta(a)(d(b)) = d(\{a, b\}) - (-1)^{|a|} \{a, d(b)\}.$$

Note that A is a DG Poisson algebra, by the graded Jacobi identity and graded Leibniz rule for bracket, we have

$$\{a, \{b, c\}\} - (-1)^{|a||b|}\{b, \{a, c\}\} = \{\{a, b\}, c\}$$

and

$$d(\{a, b\}) - (-1)^{|a|}\{a, d(b)\} = \{d(a), b\},$$

which imply that δ is a DG Lie algebra map. On the other hand, we see that

$$\begin{aligned} \delta(a)\gamma(b)(c) - (-1)^{|a||b|}\gamma(b)\delta(a)(c) &= \delta(a)(bc) - (-1)^{|a||b|}\gamma(b)\{a, c\} \\ &= \{a, bc\} - (-1)^{|a||b|}b\{a, c\} \\ &= \gamma(\{a, b\})(c) \end{aligned}$$

and

$$\begin{aligned} \gamma(a)\delta(b)(c) + (-1)^{|a||b|}\gamma(b)\delta(a)(c) &= \gamma(a)\{b, c\} + (-1)^{|a||b|}\gamma(b)\{a, c\} \\ &= a\{b, c\} + (-1)^{|a||b|}b\{a, c\} \\ &= \delta(ab)(c), \end{aligned}$$

for all $a, b, c \in A$. Hence there exists a DG algebra map ϕ from A^{ue} into $End_{\mathbb{K}}(A)$ such that $\phi\alpha = \gamma$ and $\phi\beta = \delta$. If $a \in \ker\alpha$ then $0 = \phi\alpha(a) = \gamma(a)$, and so $0 = \gamma(a)(1) = a$. It completes the proof. \square

Henceforce, we identify the DG algebra homomorphic image of a finitely generated DG Poisson algebra A under α to A and denote $\alpha(a)$ by a for all $a \in A$.

2.3. Construction of A^e . For any DG Poisson algebra A given by generators and relations, we give a “formula” for computing the universal enveloping algebra A^e of A .

2.3.1. “Anti-differential”. Let V be a graded \mathbb{K} -vector space with a homogeneous \mathbb{K} -basis $\{x_\alpha : \alpha \in \Lambda\}$ and

$$R = \frac{T(V)}{\langle x_\alpha \otimes x_\beta - (-1)^{|x_\alpha||x_\beta|}x_\beta \otimes x_\alpha \mid \forall \alpha, \beta \in \Lambda \rangle}$$

be a DG Poisson algebra with differential d and Poisson bracket $\{\cdot, \cdot\}$. Here $T(V)$ is the tensor algebra of V over \mathbb{K} and $|x|$ denotes the degree of the homogeneous element x of R .

Now for any $\alpha \in \Lambda$, we define a \mathbb{K} -linear map

$$\psi_\alpha : R \rightarrow R$$

such that

$$\psi_\alpha(x_\beta) = \delta_{\alpha\beta} \quad \text{and} \quad \psi_\alpha(ab) = a\psi_\alpha(b) + (-1)^{|a||b|}b\psi_\alpha(a),$$

for all homogeneous elements $a, b \in R$ and $\beta \in \Lambda$. The \mathbb{K} -linear map are usually called anti-differential of R .

Remark 2.8. We have the following two observations:

- It is easy to see that $|\psi_\alpha| = -|x_\alpha|$ for any $\alpha \in \Lambda$, and that the \mathbb{K} -linear map ψ_α for any $\alpha \in \Lambda$ is well-defined on R since

$$\begin{aligned} \psi_\alpha(ab - (-1)^{|a||b|}ba) &= \psi_\alpha(ab) - (-1)^{|a||b|}\psi_\alpha(ba) \\ &= a\psi_\alpha(b) + (-1)^{|a||b|}b\psi_\alpha(a) - (-1)^{|a||b|}(b\psi_\alpha(a) + (-1)^{|a||b|}a\psi_\alpha(b)) \\ &= 0. \end{aligned}$$

- If we set $\Lambda_a := \{\alpha \in \Lambda \mid \psi_\alpha(a) \neq 0\}$ for any homogeneous element $a \in R$, then Λ_a is a finite set since R obviously has a PBW-basis.

Now we give some basic properties of these “anti-differentials”.

Lemma 2.9. For any homogeneous element $a \in R$ and for all $\alpha, \beta \in \Lambda$, we have

$$(2.1) \quad \psi_\alpha(d(a)) - d\psi_\alpha(a) - \sum_{\beta \in \Lambda} (-1)^{|\psi_\beta(a)|} \psi_\beta(a) \psi_\alpha(d(x_\beta)) = 0.$$

Proof. Observe that formula (2.1) is true on any x_γ for all $\gamma \in \Lambda$ since

$$\begin{aligned} \text{LHS of (2.1)} &= \psi_\alpha(d(x_\gamma)) - d\psi_\alpha(x_\gamma) - \sum_{\beta \in \Lambda} (-1)^{|\psi_\beta(x_\gamma)|} \psi_\beta(x_\gamma) \psi_\alpha(d(x_\beta)) \\ &= \psi_\alpha(d(x_\gamma)) - \psi_\alpha(d(x_\gamma)) \\ &= 0. \end{aligned}$$

In order to prove that formula (2.1) is true for any homogeneous element $a \in R$. It suffices to prove the formula (2.1) is true for ab provided that it is true for any homogeneous elements $a, b \in R$. Thus, assume that we have the following two equations:

$$\begin{aligned} \psi_\alpha(d(a)) - d\psi_\alpha(a) - \sum_{\beta \in \Lambda} (-1)^{|\psi_\beta(a)|} \psi_\beta(a) \psi_\alpha(d(x_\beta)) &= 0, \\ \psi_\alpha(d(b)) - d\psi_\alpha(b) - \sum_{\beta \in \Lambda} (-1)^{|\psi_\beta(b)|} \psi_\beta(b) \psi_\alpha(d(x_\beta)) &= 0, \end{aligned}$$

for any homogeneous elements $a, b \in R$. We have

$$\begin{aligned} &\psi_\alpha(d(ab)) - d\psi_\alpha(ab) - \sum_{\beta \in \Lambda} (-1)^{|\psi_\beta(ab)|} \psi_\beta(ab) \psi_\alpha(d(x_\beta)) \\ &= \psi_\alpha(d(a)b + (-1)^{|a|}ad(b)) - d(a\psi_\alpha(b) + (-1)^{|a||b|}b\psi_\alpha(a)) \\ &\quad - \sum_{\beta \in \Lambda} (-1)^{|\psi_\beta(ab)|} (a\psi_\beta(b) + (-1)^{|a||b|}b\psi_\beta(a)) \psi_\alpha(d(x_\beta)) \\ &= d(a)\psi_\alpha(b) + (-1)^{(|a|+1)|b|}b\psi_\alpha(d(a)) + (-1)^{|a|}a\psi_\alpha(d(b)) + (-1)^{|a|+|a|(|b|+1)}d(b)\psi_\alpha(a) \\ &\quad - d(a)\psi_\alpha(b) - (-1)^{|a|}ad\psi_\alpha(b) - (-1)^{|a||b|}d(b)\psi_\alpha(a) - (-1)^{(|a|+1)|b|}bd\psi_\alpha(a) \\ &\quad - \sum_{\beta \in \Lambda} (-1)^{|\psi_\beta(ab)|} a\psi_\beta(b) \psi_\alpha(d(x_\beta)) - \sum_{\beta \in \Lambda} (-1)^{|\psi_\beta(ab)|+|a||b|} b\psi_\beta(a) \psi_\alpha(d(x_\beta)) \\ &= (-1)^{(|a|+1)|b|} b[\psi_\alpha(d(a)) - d\psi_\alpha(a) - \sum_{\beta \in \Lambda} (-1)^{|\psi_\beta(a)|} \psi_\beta(a) \psi_\alpha(d(x_\beta))] \\ &\quad + (-1)^{|a|} a[\psi_\alpha(d(b)) - d\psi_\alpha(b) - \sum_{\beta \in \Lambda} (-1)^{|\psi_\beta(b)|} \psi_\beta(b) \psi_\alpha(d(x_\beta))] \\ &= 0, \end{aligned}$$

as required. □

Lemma 2.10. *We have*

$$(2.2) \quad \{a, b\} = \sum_{\alpha \in \Lambda} \psi_\alpha(a) \{x_\alpha, b\},$$

for any homogeneous elements $a, b \in R$.

Proof. It is clear that formula (2.2) is true on the generators $\{x_\alpha\}_{\alpha \in \Lambda}$ of R . Thus in order to complete the proof, we only need to show

$$\{aa', bb'\} = \sum_{\alpha \in \Lambda} \psi_\alpha(aa') \{x_\alpha, bb'\}$$

provided that

$$\{a, b\} = \sum_{\alpha \in \Lambda} \psi_{\alpha}(a) \{x_{\alpha}, b\}$$

and

$$\{a', b'\} = \sum_{\alpha \in \Lambda} \psi_{\alpha}(a') \{x_{\alpha}, b'\},$$

for any homogeneous elements $a, a', b, b' \in R$.

Indeed, we have

$$\begin{aligned} \{aa', bb'\} &= \{aa', b\}b' + (-1)^{|b|(|a|+|a'|)} b\{aa', b'\} \\ &= a\{a', b\}b' + (-1)^{|a||a'|} a'\{a, b\}b' + (-1)^{|b|(|a|+|a'|)} ba\{a', b'\} + (-1)^{|b|(|a|+|a'|)+|a||a'|} ba'\{a, b'\} \\ &= \sum_{\alpha \in \Lambda} a\psi_{\alpha}(a') \{x_{\alpha}, b\}b' + (-1)^{|a||a'|} \sum_{\alpha \in \Lambda} a'\psi_{\alpha}(a) \{x_{\alpha}, b\}b' \\ &\quad + (-1)^{|b|(|a|+|a'|)} \sum_{\alpha \in \Lambda} ba\psi_{\alpha}(a') \{x_{\alpha}, b'\} + (-1)^{|b|(|a|+|a'|)+|a||a'|} \sum_{\alpha \in \Lambda} ba'\psi_{\alpha}(a) \{x_{\alpha}, b'\} \\ &= \sum_{\alpha \in \Lambda} a\psi_{\alpha}(a') \{x_{\alpha}, b\}b' + (-1)^{|b||x_{\alpha}|} \sum_{\alpha \in \Lambda} a\psi_{\alpha}(a') b\{x_{\alpha}, b'\} \\ &\quad + (-1)^{|a||a'|} \sum_{\alpha \in \Lambda} a'\psi_{\alpha}(a) \{x_{\alpha}, b\}b' + (-1)^{|b||x_{\alpha}|+|a||a'|} \sum_{\alpha \in \Lambda} a'\psi_{\alpha}(a) b\{x_{\alpha}, b'\} \\ &= \sum_{\alpha \in \Lambda} [(a\psi_{\alpha}(a') + (-1)^{|a||a'|} a'\psi_{\alpha}(a))(\{x_{\alpha}, b\}b' + (-1)^{|b||x_{\alpha}|} b\{x_{\alpha}, b'\})] \\ &= \sum_{\alpha \in \Lambda} \psi_{\alpha}(aa') \{x_{\alpha}, bb'\}, \end{aligned}$$

as required. \square

2.3.2. Now we introduce another set of indeterminates $\{y_{\alpha} \mid \alpha \in \Lambda\}$ such that $|x_{\alpha}| = |y_{\alpha}|$ for all $\alpha \in \Lambda$. Define a graded R -free algebra $F(R)$ by

$$F(R) := R\langle y_{\alpha} \mid \alpha \in \Lambda \rangle$$

and a \mathbb{k} -linear map

$$\psi : R \rightarrow F(R)$$

by

$$\psi(f) := \sum_{\alpha \in \Lambda} \psi_{\alpha}(f) y_{\alpha}$$

for any $f \in R$. Note that such ψ is well-defined since for any $f \in R$, there are only finite many $\alpha \in \Lambda$ such that $\psi_{\alpha}(f)$ is not zero. Also, let $j : R \rightarrow F(R)$ be the canonical inclusion map.

Now suppose that $(R, d, \{\cdot, \cdot\})$ is a DG Poisson algebra and I is a DG Poisson ideal of R . Put $A := R/I$, then A has a natural DG Poisson algebra structure induced from R . Let π' denote the canonical projection $\pi' : R \rightarrow A$, then π' is a DG Poisson algebra map.

As for $F(R)$, let J be the graded ideal of $F(R)$ generated by

- (1) $I, \psi(I)$,
- (2) $y_{\alpha}y_{\beta} - (-1)^{|x_{\alpha}||x_{\beta}|} y_{\beta}y_{\alpha} - \psi(\{x_{\alpha}, x_{\beta}\})$,
- (3) $y_{\alpha}x_{\beta} - (-1)^{|x_{\alpha}||x_{\beta}|} x_{\beta}y_{\alpha} - \{x_{\alpha}, x_{\beta}\}$,

where $|x_{\alpha}| = |y_{\alpha}|$ for any $\alpha \in \Lambda$, and

$$\psi(f) := \sum_{\alpha \in \Lambda} \psi_{\alpha}(f) y_{\alpha},$$

for any $f \in R$. This induces a canonical projection $\pi : F(R) \rightarrow \mathcal{A} := F(R)/J$. For the sake of simplicity, we omit the canonical projections π and π' here, and also denote by d the differential of A .

Lemma 2.11. \mathcal{A} is a DG algebra.

Proof. Note that $|\psi| = 0$, it is easy to see that \mathcal{A} is a graded algebra from its construction. Define a \mathbb{k} -linear map $\partial : \mathcal{A} \rightarrow \mathcal{A}$ by

$$\partial x_\alpha := j(d(x_\alpha)) = d(x_\alpha), \quad \partial y_\alpha := \psi(d(x_\alpha)) = \sum_{\beta \in \Lambda} \psi_\beta d(x_\alpha) y_\beta$$

for all $\alpha \in \Lambda$, and the graded Leibniz rule, i.e.,

$$\partial(ab) = \partial(a)b + (-1)^{|a|} a\partial(b)$$

for all homogeneous elements $a, b \in \mathcal{A}$. It is obvious that $|\partial| = 1$. In order to prove \mathcal{A} is a DG algebra, it suffices to prove that $\partial^2 = 0$.

Indeed, it is easy to see that $\partial(f) = d(f)$ for all $f \in A$ by using induction. We have

$$\partial^2(x_\alpha) = \partial(d(x_\alpha)) = d^2(x_\alpha) = 0,$$

for all $\alpha \in \Lambda$. Moreover, we have

$$\begin{aligned} \partial^2(y_\alpha) &= \partial\left(\sum_{\beta \in \Lambda} \psi_\beta d(x_\alpha) y_\beta\right) \\ &= \sum_{\beta \in \Lambda} [\partial\psi_\beta d(x_\alpha) y_\beta + (-1)^{|\psi_\beta d(x_\alpha)|} \psi_\beta d(x_\alpha) \partial(y_\beta)] \\ &= \sum_{\beta \in \Lambda} [d\psi_\beta d(x_\alpha) y_\beta + (-1)^{|\psi_\beta d(x_\alpha)|} \psi_\beta d(x_\alpha) \sum_{\gamma \in \Lambda} \psi_\gamma d(x_\beta) y_\gamma] \\ &= \sum_{\beta \in \Lambda} \psi_\beta d^2(x_\alpha) y_\beta - \sum_{\beta, \gamma \in \Lambda} (-1)^{|\psi_\gamma d(x_\alpha)|} \psi_\gamma d(x_\alpha) \psi_\beta d(x_\gamma) y_\beta + \sum_{\beta, \gamma \in \Lambda} (-1)^{|\psi_\beta d(x_\alpha)|} \psi_\beta d(x_\alpha) \psi_\gamma d(x_\beta) y_\gamma \\ &= - \sum_{\beta, \gamma \in \Lambda} (-1)^{|\psi_\gamma d(x_\alpha)|} \psi_\gamma d(x_\alpha) \psi_\beta d(x_\gamma) y_\beta + \sum_{\beta, \gamma \in \Lambda} (-1)^{|\psi_\beta d(x_\alpha)|} \psi_\beta d(x_\alpha) \psi_\gamma d(x_\beta) y_\gamma \\ &= 0 \end{aligned}$$

by Lemma 2.9. □

Note that there are two \mathbb{k} -linear maps $m, h : A \rightarrow \mathcal{A}$ given by

$$m(f) = f, \quad h(f) = \psi(f),$$

for all $f \in A$. We can obtain our main result in this subsection.

Theorem 2.12. (\mathcal{A}, m, h) is the universal enveloping algebra of a DG Poisson algebra A , where $A = R/I$ is defined as above.

Proof. By Lemma 2.11, (\mathcal{A}, ∂) is a DG algebra. From the construction of \mathcal{A} , there are two \mathbb{k} -linear maps $m : A \rightarrow \mathcal{A}$ sending each element $f \in A$ to f and $h : A \rightarrow \mathcal{A}$ sending each element $f \in A$ to $\psi(f)$. In fact, m is the DG algebra map and h is DG \mathbb{k} -linear map since for all homogeneous $f, g \in A$, we have

$$\begin{aligned} m(fg) &= fg = m(f)m(g), \\ \partial m(f) &= \partial(f) = d(f) = m d(f) \end{aligned}$$

and

$$\begin{aligned}
\partial h(f) &= \partial \left(\sum_{\alpha \in \Lambda} \psi_{\alpha}(f) y_{\alpha} \right) \\
&= \sum_{\alpha \in \Lambda} [\partial \psi_{\alpha}(f) y_{\alpha} + (-1)^{|\psi_{\alpha}(f)|} \psi_{\alpha}(f) \partial(y_{\alpha})] \\
&= \sum_{\alpha \in \Lambda} [d\psi_{\alpha}(f) y_{\alpha} + (-1)^{|\psi_{\alpha}(f)|} \psi_{\alpha}(f) \left(\sum_{\beta \in \Lambda} \psi_{\beta} d(x_{\alpha}) y_{\beta} \right)] \\
&= \sum_{\alpha \in \Lambda} [d\psi_{\alpha}(f) y_{\alpha}] + \sum_{\beta \in \Lambda} [\psi_{\beta} d(f) - d\psi_{\beta}(f)] y_{\beta} \\
&= h d(f)
\end{aligned}$$

by Lemma 2.9.

Notice that for any homogeneous elements $f, g \in A$, we have

$$\begin{aligned}
h(fg) &= \psi(fg) = \sum_{\alpha \in \Lambda} \psi_{\alpha}(fg) y_{\alpha} = \sum_{\alpha \in \Lambda} f \psi_{\alpha}(g) y_{\alpha} + \sum_{\alpha \in \Lambda} (-1)^{|f||g|} g \psi_{\alpha}(f) y_{\alpha} \\
&= f \psi(g) + (-1)^{|f||g|} g \psi(f) = m(f)h(g) + (-1)^{|f||g|} m(g)h(f).
\end{aligned}$$

For a monomial $f \in A$, we proceed the proof using induction on the length of f . We already know that

$$\begin{aligned}
h(x_{\alpha})m(x_{\beta}) - (-1)^{|x_{\alpha}||x_{\beta}|} m(x_{\beta})h(x_{\alpha}) &= \sum_{\gamma \in \Lambda} \psi_{\gamma}(x_{\alpha}) y_{\gamma} x_{\beta} - (-1)^{|x_{\alpha}||x_{\beta}|} x_{\beta} \sum_{\gamma \in \Lambda} \psi_{\gamma}(x_{\alpha}) y_{\gamma} \\
&= y_{\alpha} x_{\beta} - (-1)^{|x_{\alpha}||x_{\beta}|} x_{\beta} y_{\alpha} \\
&= m(\{x_{\alpha}, x_{\beta}\}).
\end{aligned}$$

For any monomials $f, g, a \in A$, assume that

$$m(\{f, a\}) = h(f)m(a) - (-1)^{|f||a|} m(a)h(f)$$

and

$$m(\{g, a\}) = h(g)m(a) - (-1)^{|g||a|} m(a)h(g).$$

Now we have

$$\begin{aligned}
&m(\{fg, a\}) \\
&= m(f\{g, a\} + (-1)^{|f||g|} g\{f, a\}) \\
&= m(f)[h(g)m(a) - (-1)^{|g||a|} m(a)h(g)] + (-1)^{|f||g|} m(g)[h(f)m(a) - (-1)^{|f||a|} m(a)h(f)] \\
&= [m(f)h(g) + (-1)^{|f||g|} m(g)h(f)]m(a) - (-1)^{|f||a|+|g||a|} m(a)[m(f)h(g) + (-1)^{|f||g|} m(g)h(f)] \\
&= h(fg)m(a) - (-1)^{|fg||a|} m(a)h(fg).
\end{aligned}$$

Applying the induction, we have

$$m(\{f, g\}) = h(f)m(g) - (-1)^{|f||g|} m(g)h(f),$$

for all $f, g \in A$. Similarly, we have $h(\{x_{\alpha}, x_{\beta}\}) = [h(x_{\alpha}), h(x_{\beta})]$. For any monomials $f, g, a \in A$, assume that

$$h(\{a, f\}) = [h(a), h(f)], \quad h(\{a, g\}) = [h(a), h(g)].$$

Then we have

$$\begin{aligned}
& h(\{a, fg\}) \\
&= h(\{a, f\}g) + (-1)^{|a||f|} f\{a, g\} \\
&= m(\{a, f\})h(g) + (-1)^{|a||f|} m(g)h(\{a, f\}) + (-1)^{|a||f|} [m(f)h(\{a, g\}) + (-1)^{|f||a,g|} m(\{a, g\})h(f)] \\
&= [h(a)m(f) - (-1)^{|a||f|} m(f)h(a)]h(g) + (-1)^{(|a|+|f|)|g|} m(g)[h(a)h(f) - (-1)^{|a||f|} h(f)h(a)] \\
&\quad + (-1)^{|a||f|} m(f)[h(a)h(g) - (-1)^{|a||g|} h(g)h(a)] + (-1)^{|f||g|} [h(a)m(g) - (-1)^{|a||g|} m(g)h(a)]h(f) \\
&= h(a)m(f)h(g) - (-1)^{|a||f|+|f||g|+|g||a|} m(g)h(f)h(a) \\
&\quad - (-1)^{|a||f|+|a||g|} m(f)h(g)h(a) + (-1)^{|f||g|} h(a)m(g)h(f) \\
&= h(a)h(fg) - (-1)^{|a||f|+|a||g|} h(fg)h(a) \\
&= [h(a), h(fg)].
\end{aligned}$$

Applying the induction, we have

$$h(\{f, g\}) = [h(f), h(g)],$$

for all $f, g \in A$.

Let (D, ∂') be a DG algebra. Let $\gamma : (A, d) \rightarrow (D, \partial')$ be a DG algebra map and let $\delta : (A, \{\cdot, \cdot\}, d) \rightarrow (D, [\cdot, \cdot], \partial')$ be a DG Lie algebra map such that

$$\begin{aligned}
\gamma(\{a, b\}) &= \delta(a)\gamma(b) - (-1)^{|a||b|}\gamma(b)\delta(a), \\
\delta(ab) &= \gamma(a)\delta(b) + (-1)^{|a||b|}\gamma(b)\delta(a),
\end{aligned}$$

for any homogeneous elements $a, b \in A$. Since $F(R)$ is a graded free algebra, there exists a graded algebra map $\phi' : F(R) \rightarrow D$ defined by

$$\phi'(x_\alpha) = \gamma(x_\alpha), \quad \phi'(y_\alpha) = \delta(x_\alpha),$$

for all $\alpha \in \Lambda$.

It is shown to be $\phi'\psi = \delta$ using induction on the length of monomials in R . Therefore we have

$$\begin{aligned}
& \phi'(I) = \gamma(I) = 0, \quad \phi'(\psi(I)) = \delta(I) = 0, \\
& \phi'(y_\alpha y_\beta - (-1)^{|x_\alpha||x_\beta|} y_\beta y_\alpha - \psi(\{x_\alpha, x_\beta\})) \\
&= \delta(x_\alpha)\delta(x_\beta) - (-1)^{|x_\alpha||x_\beta|} \delta(x_\beta)\delta(x_\alpha) - \delta(\{x_\alpha, x_\beta\}) \\
&= [\delta(x_\alpha), \delta(x_\beta)] - \delta(\{x_\alpha, x_\beta\}) = 0, \\
& \phi'(y_\alpha x_\beta - (-1)^{|x_\alpha||x_\beta|} x_\beta y_\alpha - \{x_\alpha, x_\beta\}) \\
&= \delta(x_\alpha)\gamma(x_\beta) - (-1)^{|x_\alpha||x_\beta|} \gamma(x_\beta)\delta(x_\alpha) - \gamma(\{x_\alpha, x_\beta\}) \\
&= \gamma(\{x_\alpha, x_\beta\}) - \gamma(\{x_\alpha, x_\beta\}) = 0.
\end{aligned}$$

Thus there exists a graded algebra map ϕ from \mathcal{A} into D such that $\phi m = \gamma$, $\phi h = \delta$.

Further, A graded algebra map $\phi : \mathcal{A} \rightarrow D$ is unique such that $\phi m = \gamma$, $\phi h = \delta$ by its construction and it is also a DG algebra map since for any $\alpha \in \Lambda$, we have

$$\phi \partial(x_\alpha) = \phi d(x_\alpha) = \phi m(d(x_\alpha)) = \gamma d(x_\alpha) = \partial' \gamma(x_\alpha) = \partial' \phi(x_\alpha)$$

and

$$\phi \partial(y_\alpha) = \phi \psi(d(x_\alpha)) = \phi h(d(x_\alpha)) = \delta d(x_\alpha) = \partial' \delta(x_\alpha) = \partial' \phi(y_\alpha).$$

Therefore, (\mathcal{A}, m, h) is the universal enveloping algebra of a DG Poisson algebra A , as required. \square

Remark 2.13. From now on, we let $A^e := \mathcal{A}$ denote the universal enveloping algebra of a DG Poisson algebra A given by generators and relations.

3. POINCARÉ-BIRKHOFF-WITT THEOREM FOR UNIVERSAL ENVELOPING ALGEBRAS

In this section, we develop an algorithm to find a \mathbb{k} -linear basis of the universal enveloping algebra A^e .

3.1. Gröbner-Shirshov basis theory. In this subsection, we recall the Gröbner-Shirshov basis theory developed in [8] and [9].

Let $X = \{x_1, x_2, \dots\}$ be a set of alphabets indexed by positive integers. Define a linear ordering on X $<$ by setting $x_i < x_j$ if and only if $i < j$. Let X^* be the free monoid of associative monomials on X . We denote the empty monomial by 1 and the length of a monomial u by $l(u)$ with $l(1) = 0$. We consider two linear ordering $<$ and \ll on X^* defined as follows [9]:

- (i) $1 < u$ for any nonempty monomial u ; and inductively, $u < v$ whenever $u = x_i u'$, $v = x_j v'$ and $x_i < x_j$ or $x_i = x_j$ and $u' < v'$ with $x_i, x_j \in X$,
- (ii) $u \ll v$ if $l(u) < l(v)$ or $l(u) = l(v)$ and $u < v$.

The ordering $<$ (resp. \ll) is called the lexicographic ordering (resp. degree-lexicographic ordering). It is easy to see \ll is a monomial order on X^* , that is, $x \ll y$ implies $axb \ll ayb$ for all $a, b \in X^*$.

Let T_X be the DG free k -algebra generated by X , let I be a DG ideal of T_X and let $T_0 = T_X/I$, then T_0 is a DG algebra. The image of $p \in T_X$ in T_0 under the canonical DG quotient map will also be denoted by p . Given a nonzero element $p \in T_0$, we denote by \bar{p} the maximal monomial appearing in p under the ordering \ll . Thus $p = \alpha \bar{p} + \sum \beta_i w_i$ with $\alpha, \beta_i \in \mathbb{k}$, $w_i \in X^*$, $\alpha \neq 0$ and $w_i \ll \bar{p}$. The α is called the leading coefficient of p and if $\alpha = 1$, then p is said to be monic. Recall that the composition of p and q as follows.

Definition 3.1. [8] Let p and q be monic elements of T_0 .

- (a) If there exist a and b in X^* such that $\bar{p}a = b\bar{q} = w$ with $l(\bar{p}) > l(b)$, then the composition of intersection is defined to be $(p, q)_w = pa - bq$.
- (b) If there exist a and b in X^* such that $a \neq 1$, $a\bar{p}b = \bar{q} = w$, then the composition of inclusion is defined to be $(p, q)_w = apb - q$.

Let S be a subset of monic elements of T_0 and let J be the DG ideal of T_0 generated by S . Then we say the DG algebra T_0/J is defined by S . The image of $p \in T_0$ in T_0/J under the canonical quotient maps will also be denoted by p as long as there is no peril of confusion. Next, let $p, q \in T_0$ and $w \in X^*$. We define a congruence relation on T_0 as follows: $p \equiv q \pmod{(J; w)}$ if and only if $p - q = \sum \alpha_i a_i s_i b_i$, where $\alpha_i \in \mathbb{k}$, $a_i, b_i \in X^*$, $s_i \in S$ and $a_i \bar{s}_i b_i \ll w$. A set S of monic elements of T_0 is said to be closed under the composition if for any $p, q \in S$ and $w \in X^*$ such that $(p, q)_w$ is defined, we have $(p, q)_w \equiv 0 \pmod{(J; w)}$.

Now we introduce another definition of monomials and then prove the following generalization of Shirshov's Composition Lemma to the representations of DG associative algebras.

Definition 3.2. [8] A monomial $u \in X^*$ is said to be S -standard in T_0 if $u \neq a\bar{s}b$ for any $s \in S$ and $a, b \in X^*$. Otherwise, the monomial u is said to be S -reducible in T_0 .

Theorem 3.3. Let S be a subset of monic elements in T_0 and let T_0/J be the DG algebra is defined by S . If S is closed under composition in T_0 and the image of $p \in T_0$ is zero in T_0/J , then the monomial \bar{p} is S -reducible in T_0 .

Proof. Since the image of $p \in T_0$ is zero in T_0/J , we have $p = \sum \alpha_i a_i s_i b_i$, where $\alpha_i \in \mathbb{k}$, $a_i, b_i \in X^*$ and $s_i \in S$. Choose the maximal monomial w in the degree-lexicographic ordering \ll among the monomials $\{a_i \overline{s_i} b_i\}$ in the expression of p . If $\bar{p} = w$, then we are done. Suppose this is not the case, then $\bar{p} \ll w$ and without loss of generality, we may assume that the following case hold: $w = a_1 \overline{s_1} b_1 = a_2 \overline{s_2} b_2$.

If $w = a_1 \overline{s_1} b_1 = a_2 \overline{s_2} b_2$, then we should show that $a_1 s_1 b_1 \equiv a_2 s_2 b_2 \pmod{(J; w)}$. There are three possibilities:

(i) If the monomials $\overline{s_1}$ and $\overline{s_2}$ have empty intersection in w , then we may assume that $a_1 s_1 b_1 = a s_1 b \overline{s_2} c$ and $a_2 s_2 b_2 = a \overline{s_1} b s_2 c$, where $a, b, c \in X^*$. Thus

$$a_2 s_2 b_2 - a_1 s_1 b_1 = -a(s_1 - \overline{s_1}) b s_2 c + a s_1 b(s_2 - \overline{s_2}) c,$$

which implies $a_1 s_1 b_1 \equiv a_2 s_2 b_2 \pmod{(J; w)}$.

(ii) If $\overline{s_1} = u_1 u_2$ and $\overline{s_2} = u_2 u_3$ for some $u_2 \neq 1$, then $a_2 = a_1 u_1$, $b_1 = u_3 b_2$ and

$$\begin{aligned} a_2 s_2 b_2 - a_1 s_1 b_1 &= a_1 u_1 s_2 b_2 - a_1 s_1 u_3 b_2 \\ &= -a_1(s_1 u_3 - u_1 s_2) b_2 = a_1(s_1, s_2)_{u_1 u_2 u_3} b_2. \end{aligned}$$

Since $\overline{(s_1, s_2)_{u_1 u_2 u_3}} \ll u_1 u_2 u_3$ and S is closed under composition in T_0 , we obtain $a_1 s_1 b_1 \equiv a_2 s_2 b_2 \pmod{(J; w)}$.

(iii) If $\overline{s_1} = u_1 \overline{s_2} u_2$, then $a_2 = a_1 u_1$, $b_2 = u_2 b_1$ and

$$\begin{aligned} a_2 s_2 b_2 - a_1 s_1 b_1 &= a_1 u_1 s_2 u_2 b_1 - a_1 s_1 b_1 \\ &= a_1(u_1 s_2 u_2 - s_1) b_1 = a_1(s_2, s_1)_{\overline{s_1}} b_1. \end{aligned}$$

Since $\overline{(s_2, s_1)_{\overline{s_1}}} \ll \overline{s_1}$ and S is closed under composition in T_0 , we get $a_1 s_1 b_1 \equiv a_2 s_2 b_2 \pmod{(J; w)}$.

Therefore, p can be written as $p = \sum \alpha'_i a'_i s'_i b'_i$, where $a'_i \overline{s'_i} b'_i \ll w$ for all i . Choose the maximal monomial w_1 in the ordering \ll among $\{a'_i \overline{s'_i} b'_i\}$. If $\bar{p} = w_1$, then we are done. If this is not the case, repeat the above process. Since X is indexed by the set of positive integers, this process must terminate in finite steps, which completes the proof. \square

As a corollary, we obtain:

Proposition 3.4. Let $\mathcal{A} \subseteq X^*$ form a \mathbb{k} -linear basis of DG algebra $T_0 = T_X/I$, let S be a subset of monic elements of T_0 and let T_0/J be the DG algebra is defined by S . Then the following are equivalent:

- (i) S is closed under composition in T_0 .
- (ii) the subset of \mathcal{A} consisting of S -standard monomials in T_0 forms a \mathbb{k} -linear basis of the DG algebra T_0/J .

Proof. Copy the proof of proposition 1.9 in [8]. \square

3.2. PBW-basis for the universal enveloping algebra. Let A be a DG Poisson homomorphic image of a DG Poisson algebra R with an arbitrary differential d and Poisson bracket $\{\cdot, \cdot\}$, where

$$R = \frac{T(V)}{\langle x_\alpha \otimes x_\beta - (-1)^{|x_\alpha||x_\beta|} x_\beta \otimes x_\alpha \mid \forall \alpha, \beta \in \Lambda \rangle}$$

and Λ is a finite index set. For the convenience of the narrative, assume that $\Lambda = \{1, 2, \dots, n\}$. In this subsection, we will find a \mathbb{k} -linear basis of R^e and develop an algorithm to find a \mathbb{k} -linear basis of the universal enveloping algebra A^e . From now on, assume that,

- $(R, \{\cdot, \cdot\}, d)$ is a DG Poisson algebra, where R is defined as above and Λ is a finite index set. Assume that $\Lambda = \{1, 2, \dots, n\}$.
- $A = R/I$, where I is a DG Poisson ideal of R .
- $F(R) = R\langle y_i \mid i \in \Lambda \rangle$.

- $\psi : R \rightarrow F(R)$, $\psi(f) = \sum_{r=1}^n \psi_r(f)y_r$.
- J is the graded ideal of $F(R)$ generated

$$I, \psi(I), y_i x_j - (-1)^{|x_i||x_j|} x_j y_i - \{x_i, x_j\}, y_i y_j - (-1)^{|x_i||x_j|} y_j y_i - \psi(\{x_i, x_j\})$$

for all $i, j = 1, 2, \dots, n$.

- $A^e = F(R)/J$ is the universal enveloping algebra of A and let $\pi : F(R) \rightarrow A^e$ denote the canonical projection.

Now define a linear ordering on the index set $\Lambda \times \Lambda$ by

$$(l, m) < (r, s) \iff m < s \text{ or } m = s \text{ and } l < r$$

and a grading on $F(R)$ by

$$\deg(x_i) = (1, 0), \deg(y_i) = (0, 1)$$

for all $i \in \Lambda$, hence the grading of a monomial $u = u_1 \cdots u_l \in F(R)$, where $u_j = x_i$ or $u_j = y_i$, is defined by

$$\deg(u) = \deg(u_1) + \cdots + \deg(u_l) \in \Lambda \times \Lambda.$$

We give an ordering $<$ on the set of generators of $F(R)$ by

$$x_1 < x_2 < \cdots < x_n < y_1 < y_2 < \cdots < y_n.$$

Therefore there is a well-ordering $<$ on the set of all monomials in $F(R)$. That is, for monomials $u = u_1 \cdots u_l$ and $v = v_1 \cdots v_m$, we denote $u < v$ if one of the following conditions holds:

- (i) $\deg(u) < \deg(v)$.
- (ii) $\deg(u) = \deg(v)$, $u_1 = v_1, \dots, u_r = v_r$ and $u_{r+1} < v_{r+1}$ for some $r \in \Lambda$.

Note that the ordering $<$ is a monomial order.

Example 3.5. For $u = x_2 x_3 y_3^2 y_1$, $v = x_3 y_2 x_1^2$ and $w = x_2 y_3 x_3 y_2 y_1$, we have $\deg(u) = \deg(w) = (2, 3)$ and $\deg(v) = (3, 1)$, hence $v < u < w$.

Lemma 3.6. For a monomial $f = x_{i_1} \cdots x_{i_r} \in F(R)$, $y_i f \equiv (-1)^{|x_i||f|} f y_i + \{x_i, f\} \mod(J; y_i f)$.

Proof. We proceed the proof using induction on $l(f) = r$. If $r = 0$ or 1 , then it is trivial. Assume that the statement is true for monomials with length less than r . Then we have that

$$\begin{aligned} y_i f &= y_i (x_{i_1} \cdots x_{i_{r-1}}) x_{i_r} \\ &\equiv ((-1)^{|x_i||x_{i_1} \cdots x_{i_{r-1}}|} x_{i_1} \cdots x_{i_{r-1}} y_i + \{x_i, x_{i_1} \cdots x_{i_{r-1}}\}) x_{i_r} \\ &\equiv (-1)^{|x_i||x_{i_1} \cdots x_{i_{r-1}}|} x_{i_1} \cdots x_{i_{r-1}} ((-1)^{|x_i||x_{i_r}|} x_{i_r} y_i + \{x_i, x_{i_r}\}) + \{x_i, x_{i_1} \cdots x_{i_{r-1}}\} x_{i_r} \\ &\equiv (-1)^{|x_i||f|} f y_i + \{x_i, f\} \mod(J; y_i f) \end{aligned}$$

by the induction hypothesis and biderivation property. □

Lemma 3.7. For a monomial $f = x_{i_1} \cdots x_{i_r} \in F(R)$,

$$\psi(f) x_i \equiv (-1)^{|x_i||f|} x_i \psi(f) + \{f, x_i\} \mod(J; \overline{\psi(f)} x_i).$$

Proof. By Lemma 2.10 and the definition of ψ , we have $\{f, x_i\} = \sum_s \psi_s(f)\{x_s, x_i\}$ and $\psi(f) = \sum_s \psi_s(f)y_s$. Thus

$$\begin{aligned}
\psi(f)x_i &= \sum_s \psi_s(f)y_s x_i \\
&\equiv \sum_s \psi_s(f)((-1)^{|x_s||x_i|}x_i y_s + \{x_s, x_i\}) \\
&\equiv \sum_s (-1)^{|x_s||x_i|}(-1)^{(|f|-|x_s|)|x_i|}x_i \psi_s(f)y_s + \sum_s \psi_s(f)\{x_s, x_i\} \\
&\equiv (-1)^{|f||x_i|}x_i \psi(f) + \sum_s \psi_s(f)\{x_s, x_i\} \\
&\equiv (-1)^{|f||x_i|}x_i \psi(f) + \{f, x_i\} \bmod(J; \overline{\psi(f)}x_i)
\end{aligned}$$

by Lemma 3.6. □

Lemma 3.8. For a monomial $f = x_{i_1} \cdots x_{i_r} \in F(R)$,

$$\begin{aligned}
y_i \psi(f) &\equiv (-1)^{|x_i||f|} \psi(f)y_i + \sum_{s,t} (-1)^{|x_i||\psi_s(f)|} \psi_s(f)\psi_t(f_{is})y_t \\
&\quad + \sum_{s,t} (-1)^{|x_i||\psi_t\psi_s(f)|} \psi_t\psi_s(f)f_{it}y_s \bmod(J; y_i \overline{\psi(f)}),
\end{aligned}$$

where $f_{ij} = \{x_i, x_j\}$.

Proof. Note that $\psi(f) = \sum_s \psi_s(f)y_s$ and $\{x_i, f\} = \sum_s (-1)^{|x_i||\psi_s(f)|} \psi_s(f)f_{is}$, we have

$$\begin{aligned}
y_i \psi(f) &= y_i \sum_s \psi_s(f)y_s \\
&\equiv \sum_s ((-1)^{|\psi_s(f)||x_i|} \psi_s(f)y_i + \{x_i, \psi_s(f)\})y_s \\
&\equiv \sum_s (-1)^{|\psi_s(f)||x_i|} \psi_s(f)((-1)^{|x_i||x_s|}y_s y_i + \psi(f_{is})) + \sum_s \{x_i, \psi_s(f)\}y_s \\
&\equiv (-1)^{|x_i||f|} \psi(f)y_i + \sum_{s,t} (-1)^{|x_i||\psi_s(f)|} \psi_s(f)\psi_t(f_{is})y_t \\
&\quad + \sum_{s,t} (-1)^{|x_i||\psi_t\psi_s(f)|} \psi_t\psi_s(f)f_{it}y_s \bmod(J; y_i \overline{\psi(f)})
\end{aligned}$$

by Lemma 3.6. □

Lemma 3.9. Retain the above notions, we have that

$$\begin{aligned}
&\sum_{s,t} (-1)^{|x_i||\psi_s(f_{jk})|} [\psi_s(f_{jk})\psi_t(f_{is}) + (-1)^{|\psi_s(f_{jk})||f_{is}|} f_{is}\psi_t\psi_s(f_{jk})]y_t \\
&+ \sum_{s,t} (-1)^{|x_k||f_{ij}|+|x_k||\psi_s(f_{ij})|} [\psi_s(f_{ij})\psi_t(f_{ks}) + (-1)^{|\psi_s(f_{ij})||f_{ks}|} f_{ks}\psi_t\psi_s(f_{ij})]y_t \\
&+ \sum_{s,t} (-1)^{|x_i||f_{jk}|+|x_j||\psi_s(f_{ki})|} [\psi_s(f_{ki})\psi_t(f_{js}) + (-1)^{|\psi_s(f_{ki})||f_{js}|} f_{js}\psi_t\psi_s(f_{ki})]y_t \equiv 0 \bmod J,
\end{aligned}$$

where $f_{ij} = \{x_i, x_j\}$.

Proof. Since $\{f, x_i\} = \sum_s \psi_s(f)\{x_s, x_i\}$, we have

$$\begin{aligned} 0 &\equiv \{x_i, f_{jk}\} + (-1)^{|x_k||f_{ij}|}\{x_k, f_{ij}\} + (-1)^{|x_i||f_{jk}|}\{x_j, f_{ki}\} \\ &\equiv \sum_s (-1)^{|x_i||\psi_s(f_{jk})|}\psi_s(f_{jk})f_{is} + (-1)^{|x_k||f_{ij}|}\sum_s (-1)^{|x_k||\psi_s(f_{ij})|}\psi_s(f_{ij})f_{ks} \\ &\quad + (-1)^{|x_i||f_{jk}|}\sum_s (-1)^{|x_j||\psi_s(f_{ki})|}\psi_s(f_{ki})f_{js}. \end{aligned}$$

Hence

$$\begin{aligned} 0 &\equiv \psi\left[\sum_s (-1)^{|x_i||\psi_s(f_{jk})|}\psi_s(f_{jk})f_{is} + (-1)^{|x_k||f_{ij}|}\sum_s (-1)^{|x_k||\psi_s(f_{ij})|}\psi_s(f_{ij})f_{ks}\right] \\ &\quad + \psi\left[(-1)^{|x_i||f_{jk}|}\sum_s (-1)^{|x_j||\psi_s(f_{ki})|}\psi_s(f_{ki})f_{js}\right] \\ &= \sum_s (-1)^{|x_i||\psi_s(f_{jk})|}\sum_t \psi_t(\psi_s(f_{jk})f_{is})y_t + (-1)^{|x_k||f_{ij}|}\sum_s (-1)^{|x_k||\psi_s(f_{ij})|}\sum_t \psi_t(\psi_s(f_{ij})f_{ks})y_t \\ &\quad + (-1)^{|x_i||f_{jk}|}\sum_s (-1)^{|x_j||\psi_s(f_{ki})|}\sum_t \psi_t(\psi_s(f_{ki})f_{js})y_t \\ &= \sum_{s,t} (-1)^{|x_i||\psi_s(f_{jk})|}[\psi_s(f_{jk})\psi_t(f_{is}) + (-1)^{|\psi_s(f_{jk})||f_{is}|}f_{is}\psi_t\psi_s(f_{jk})]y_t \\ &\quad + \sum_{s,t} (-1)^{|x_k||f_{ij}|+|x_k||\psi_s(f_{ij})|}[\psi_s(f_{ij})\psi_t(f_{ks}) + (-1)^{|\psi_s(f_{ij})||f_{ks}|}f_{ks}\psi_t\psi_s(f_{ij})]y_t \\ &\quad + \sum_{s,t} (-1)^{|x_i||f_{jk}|+|x_j||\psi_s(f_{ki})|}[\psi_s(f_{ki})\psi_t(f_{js}) + (-1)^{|\psi_s(f_{ki})||f_{js}|}f_{js}\psi_t\psi_s(f_{ki})]y_t \mod J. \end{aligned}$$

□

Lemma 3.10. For a monomial $f = x_{i_1} \cdots x_{i_r} \in F(R)$,

$$\sum_{s,t} (-1)^{|x_s||x_t|}\psi_s\psi_t(f)f_{it}y_s = \sum_{s,t} \psi_t\psi_s(f)f_{it}y_s,$$

where $f_{ij} = \{x_i, x_j\}$.

Proof. We proceed the proof using induction on $l(f) = r$. If $r = 1$ or 2 , then it is trivial. Assume that the statement is true for monomials with length less than r . Set $f_{ij} = \{x_i, x_j\}$, then we have that

$$\begin{aligned} &\sum_{s,t} (-1)^{|x_s||x_t|}\psi_s\psi_t(x_{i_1} \cdots x_{i_{r-1}} \cdot x_{i_r})f_{it}y_s \\ &= \sum_{s,t} (-1)^{|x_s||x_t|}\psi_s[x_{i_1} \cdots x_{i_{r-1}}\psi_t(x_{i_r}) + (-1)^{|x_{i_r}||x_{i_1} \cdots x_{i_{r-1}}|}x_{i_r}\psi_t(x_{i_1} \cdots x_{i_{r-1}})]f_{it}y_s \\ &= \sum_{s,t} (-1)^{|x_s||x_t|}[x_{i_1} \cdots x_{i_{r-1}}\psi_s\psi_t(x_{i_r}) + (-1)^{|x_{i_1} \cdots x_{i_{r-1}}||\psi_t(x_{i_r})|}\psi_t(x_{i_r})\psi_s(x_{i_1} \cdots x_{i_{r-1}}) \\ &\quad + (-1)^{|x_{i_r}||x_{i_1} \cdots x_{i_{r-1}}|}(x_{i_r}\psi_s\psi_t(x_{i_1} \cdots x_{i_{r-1}}) + (-1)^{|x_{i_r}||\psi_t(x_{i_1} \cdots x_{i_{r-1}})|}\psi_t(x_{i_1} \cdots x_{i_{r-1}})\psi_s(x_{i_r}))]f_{it}y_s \end{aligned}$$

and

$$\begin{aligned}
& \sum_{s,t} \psi_t \psi_s(x_{i_1} \cdots x_{i_{r-1}} \cdot x_{i_r}) f_{it} y_s \\
&= \sum_{s,t} \psi_t [x_{i_1} \cdots x_{i_{r-1}} \psi_s(x_{i_r}) + (-1)^{|x_{i_r}| |x_{i_1} \cdots x_{i_{r-1}}|} x_{i_r} \psi_s(x_{i_1} \cdots x_{i_{r-1}})] f_{it} y_s \\
&= \sum_{s,t} [x_{i_1} \cdots x_{i_{r-1}} \psi_t \psi_s(x_{i_r}) + (-1)^{|x_{i_1} \cdots x_{i_{r-1}}| |\psi_s(x_{i_r})|} \psi_s(x_{i_r}) \psi_t(x_{i_1} \cdots x_{i_{r-1}})] f_{it} y_s \\
&\quad + \sum_{s,t} (-1)^{|x_{i_r}| |x_{i_1} \cdots x_{i_{r-1}}|} [x_{i_r} \psi_t \psi_s(x_{i_1} \cdots x_{i_{r-1}}) + (-1)^{|x_{i_r}| |\psi_s(x_{i_1} \cdots x_{i_{r-1}})|} \psi_s(x_{i_1} \cdots x_{i_{r-1}}) \psi_t(x_{i_r})] f_{it} y_s.
\end{aligned}$$

Thus

$$\sum_{s,t} (-1)^{|x_s| |x_t|} \psi_s \psi_t(x_{i_1} \cdots x_{i_{r-1}} \cdot x_{i_r}) f_{it} y_s = \sum_{s,t} \psi_t \psi_s(x_{i_1} \cdots x_{i_{r-1}} \cdot x_{i_r}) f_{it} y_s$$

by the induction hypothesis. Therefore, we finish the proof. \square

Lemma 3.11. *In (A^e, m, h) , we have*

- (a) $y_i f = (-1)^{|x_i| |f|} f y_i + \{x_i, f\}$ for $f \in A$,
- (b) $h(f) x_i = (-1)^{|x_i| |f|} x_i h(f) + \{f, x_i\}$ for $f \in A$,
- (c) $y_i h(f) = (-1)^{|x_i| |f|} h(f) y_i + h(\{x_i, f\})$ for $f \in A$.

Proof. From Lemmas 3.6 and 3.7, it is easy to see (a) and (b). Since $\psi(f) = \sum_t \psi_t(f) y_t$ and $\{x_i, f\} = \sum_s (-1)^{|x_i| |\psi_s(f)|} \psi_s(f) \{x_i, x_s\}$ for any $f \in R$, we have that

$$\begin{aligned}
\psi(\{x_i, f\}) &= \sum_t \psi_t(\{x_i, f\}) y_t \\
&\equiv \sum_t \psi_t \left(\sum_s (-1)^{|x_i| |\psi_s(f)|} \psi_s(f) f_{is} \right) y_t \\
&\equiv \sum_{s,t} (-1)^{|x_i| |\psi_s(f)|} \psi_s(f) \psi_t(f_{is}) y_t + \sum_{s,t} (-1)^{|\psi_s(f)| (|x_i| + |f_{is}|)} f_{is} \psi_t \psi_s(f) y_t \\
&\equiv \sum_{s,t} (-1)^{|x_i| |\psi_s(f)|} \psi_s(f) \psi_t(f_{is}) y_t + \sum_{s,t} (-1)^{|x_i| |\psi_t \psi_s(f)|} \psi_t \psi_s(f) f_{it} y_s \mod J
\end{aligned}$$

by Lemma 3.10, where $f_{ij} = \{x_i, x_j\}$. Hence (c) follows from Lemma 3.8. \square

Theorem 3.12. *Let $R = T(V)/\langle x_\alpha \otimes x_\beta - (-1)^{|x_\alpha| |x_\beta|} x_\beta \otimes x_\alpha \mid \forall \alpha, \beta \in \Lambda \rangle$ be a DG Poisson algebra with an arbitrary differential d and Poisson structure $\{\cdot, \cdot\}$. Then the universal enveloping algebra R^e has a \mathbb{k} -linear basis*

$$\mathfrak{B} = \{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n} \mid i_r, j_r = 0, 1, 2, \dots\}.$$

Proof. Since

$$R = \frac{T(V)}{\langle x_i \otimes x_j - (-1)^{|x_i| |x_j|} x_j \otimes x_i \mid \forall i, j \in \Lambda \rangle}$$

and $\psi(x_i x_j - (-1)^{|x_i| |x_j|} x_j x_i) = 0$, the universal enveloping algebra R^e is $R^e = F(R)/J'$, where J' is the DG ideal generated by

- (1) $x_{ij} : x_i x_j - (-1)^{|x_i| |x_j|} x_j x_i$,
- (2) $y_{ij} : y_i y_j - (-1)^{|x_i| |x_j|} y_j y_i - \psi(\{x_i, x_j\})$,
- (3) $z_{ij} : y_i x_j - (-1)^{|x_i| |x_j|} x_j y_i - \{x_i, x_j\}$,

for all i, j . By Proposition 3.4, it is enough to show that the generators of J' is closed under composition in $F(R)$. There are only four possible compositions among the generators of J' :

- $(x_{ij}, x_{jk})_{x_i x_j x_k} (i > j > k)$
- $(y_{ij}, y_{jk})_{y_i y_j y_k} (i > j > k)$
- $(y_{ij}, z_{jk})_{y_i y_j x_k} (i > j)$
- $(z_{ij}, x_{jk})_{y_i x_j x_k} (j > k)$

Case1. $(x_{ij}, x_{jk})_{x_i x_j x_k} (i > j > k)$

$$\begin{aligned}
 (x_{ij}, x_{jk})_{x_i x_j x_k} &= x_{ij} x_k - x_i x_{jk} \\
 &= (x_i x_j - (-1)^{|x_i||x_j|} x_j x_i) x_k - x_i (x_j x_k - (-1)^{|x_j||x_k|} x_k x_j) \\
 &= -(-1)^{|x_i||x_j|} x_j x_i x_k + (-1)^{|x_j||x_k|} x_i x_k x_j \\
 &\equiv -(-1)^{2|x_i||x_j|} x_i x_j x_k + (-1)^{2|x_j||x_k|} x_i x_j x_k \equiv 0 \pmod{J; x_i x_j x_k}.
 \end{aligned}$$

Case2. $(y_{ij}, y_{jk})_{y_i y_j y_k} (i > j > k)$

Set $\{x_i, x_j\} = f_{ij}$. Since $\psi(f) = \sum_s \psi_s(f) y_s$ and $\{x_i, f\} = \sum_s (-1)^{|x_i||\psi_s(f)|} \psi_s(f) f_{is}$, we have

$$\begin{aligned}
 (y_{ij}, y_{jk})_{y_i y_j y_k} &= y_{ij} y_k - y_i y_{jk} \\
 &= [y_i y_j - (-1)^{|x_i||x_j|} y_j y_i - \psi(f_{ij})] y_k - y_i [y_j y_k - (-1)^{|x_j||x_k|} y_k y_j - \psi(f_{jk})] \\
 &\equiv -(-1)^{|x_i||x_j|} y_j [(-1)^{|x_i||x_k|} y_k y_i + \psi(f_{ik})] - \psi(f_{ij}) y_k \\
 &\quad + (-1)^{|x_j||x_k|} [(-1)^{|x_i||x_k|} y_k y_i + \psi(f_{ik})] y_j + y_i \psi(f_{jk}) \\
 &\equiv -(-1)^{|x_i||x_j|+|x_i||x_k|} [(-1)^{|x_j||x_k|} y_k y_j + \psi(f_{jk})] y_i - (-1)^{|x_i||x_j|} y_j \psi(f_{ik}) - \psi(f_{ij}) y_k \\
 &\quad + (-1)^{|x_i||x_k|+|x_j||x_k|} y_k [(-1)^{|x_i||x_j|} y_j y_i + \psi(f_{ij})] + (-1)^{|x_j||x_k|} \psi(f_{ik}) y_j + y_i \psi(f_{jk}) \\
 &\equiv [\sum_{s,t} (-1)^{|x_i||\psi_s(f_{jk})|} \psi_s(f_{jk}) \psi_t(f_{is}) y_t + \sum_{s,t} (-1)^{|x_i||\psi_t \psi_s(f_{jk})|} \psi_t \psi_s(f_{jk}) f_{it} y_s] \\
 &\quad + (-1)^{|x_i||f_{jk}|} [\sum_{s,t} (-1)^{|x_j||\psi_s(f_{ki})|} \psi_s(f_{ki}) \psi_t(f_{js}) y_t + \sum_{s,t} (-1)^{|x_j||\psi_t \psi_s(f_{ki})|} \psi_t \psi_s(f_{ki}) f_{jt} y_s] \\
 &\quad + (-1)^{|x_k||f_{ij}|} [\sum_{s,t} (-1)^{|x_k||\psi_s(f_{ij})|} \psi_s(f_{ij}) \psi_t(f_{ks}) y_t + \sum_{s,t} (-1)^{|x_k||\psi_t \psi_s(f_{ij})|} \psi_t \psi_s(f_{ij}) f_{kt} y_s]
 \end{aligned}$$

by Lemma 3.8. It is easy to see that

$$\sum_{s,t} (-1)^{|x_k||\psi_t \psi_s(f_{ij})|} \psi_t \psi_s(f_{ij}) f_{kt} y_s \equiv \sum_{s,t} (-1)^{|x_s||\psi_s(f_{ij})|} f_{ks} \psi_t \psi_s(f_{ij}) y_t \pmod{J}$$

by Lemma 3.10.

Hence

$$\begin{aligned}
 (y_{ij}, y_{jk})_{y_i y_j y_k} &\equiv \sum_{s,t} (-1)^{|x_i||\psi_s(f_{jk})|} [\psi_s(f_{jk}) \psi_t(f_{is}) + (-1)^{|\psi_s(f_{jk})||f_{is}|} f_{is} \psi_t \psi_s(f_{jk})] y_t \\
 &\quad + \sum_{s,t} (-1)^{|x_k||f_{ij}|+|x_k||\psi_s(f_{ij})|} [\psi_s(f_{ij}) \psi_t(f_{ks}) + (-1)^{|\psi_s(f_{ij})||f_{ks}|} f_{ks} \psi_t \psi_s(f_{ij})] y_t \\
 &\quad + \sum_{s,t} (-1)^{|x_i||f_{jk}|+|x_j||\psi_s(f_{ki})|} [\psi_s(f_{ki}) \psi_t(f_{js}) + (-1)^{|\psi_s(f_{ki})||f_{js}|} f_{js} \psi_t \psi_s(f_{ki})] y_t \\
 &\equiv 0 \pmod{J; y_i y_j y_k}
 \end{aligned}$$

by Lemma 3.9.

Case3. $(y_{ij}, z_{jk})_{y_i y_j x_k} (i > j)$

$$\begin{aligned}
(y_{ij}, z_{jk})_{y_i y_j x_k} &= y_{ij} x_k - y_i z_{jk} \\
&= (y_i y_j - (-1)^{|x_i||x_j|} y_j y_i - \psi(\{x_i, x_j\})) x_k - y_i (y_j x_k - (-1)^{|x_j||x_k|} x_k y_j - \{x_j, x_k\}) \\
&\equiv -(-1)^{|x_i||x_j|} y_j ((-1)^{|x_i||x_k|} x_k y_i + \{x_i, x_k\}) + (-1)^{|x_j||x_k|} ((-1)^{|x_i||x_k|} x_k y_i + \{x_i, x_k\}) y_j \\
&\quad - \psi(\{x_i, x_j\}) x_k + y_i \{x_j, x_k\} \\
&\equiv -(-1)^{|x_i||x_j|+|x_i||x_k|} ((-1)^{|x_j||x_k|} x_k y_j + \{x_j, x_k\}) y_i - (-1)^{|x_i||x_j|} y_j \{x_i, x_k\} - \psi(\{x_i, x_j\}) x_k \\
&\quad + (-1)^{|x_j||x_k|+|x_i||x_k|} x_k ((-1)^{|x_i||x_j|} y_j y_i + \psi(\{x_i, x_j\})) + (-1)^{|x_j||x_k|} \{x_i, x_k\} y_j + y_i \{x_j, x_k\} \\
&\equiv -(-1)^{|x_i||x_j|+|x_i||x_k|} \{x_j, x_k\} y_i - (-1)^{|x_i||x_j|} ((-1)^{|x_j||x_k|} \{x_i, x_k\} y_j + \{x_j, \{x_i, x_k\}\}) \\
&\quad - ((-1)^{|x_k||\{x_i, x_j\}|} x_k \psi(\{x_i, x_j\}) + \{\{x_i, x_j\}, x_k\}) + (-1)^{|x_j||x_k|} \{x_i, x_k\} y_j \\
&\quad + (-1)^{|x_j||x_k|+|x_i||x_k|} x_k \psi(\{x_i, x_j\}) + (-1)^{|x_i||\{x_j, x_k\}|} \{x_j, x_k\} y_i + \{x_i, \{x_j, x_k\}\} \\
&\equiv -(-1)^{|x_i||x_j|} \{x_j, \{x_i, x_k\}\} - \{\{x_i, x_j\}, x_k\} + \{x_i, \{x_j, x_k\}\} \\
&\equiv 0 \mod(J; y_i y_j x_k)
\end{aligned}$$

by the graded Jacobi identity, Lemmas 3.6 and 3.7 .

Case4. $(z_{ij}, x_{jk})_{y_i x_j x_k} (j > k)$

$$\begin{aligned}
(z_{ij}, x_{jk})_{y_i x_j x_k} &= z_{ij} x_k - y_i x_{jk} \\
&= (y_i x_j - (-1)^{|x_i||x_j|} x_j y_i - \{x_i, x_j\}) x_k - y_i (x_j x_k - (-1)^{|x_j||x_k|} x_k x_j) \\
&\equiv -(-1)^{|x_i||x_j|} x_j ((-1)^{|x_i||x_k|} x_k y_i + \{x_i, x_k\}) \\
&\quad - \{x_i, x_j\} x_k + (-1)^{|x_j||x_k|} ((-1)^{|x_i||x_k|} x_k y_i + \{x_i, x_k\}) x_j \\
&\equiv -(-1)^{|x_i||x_j|+|x_i||x_k|+|x_j||x_k|} x_k x_j y_i - (-1)^{|x_i||x_j|} x_j \{x_i, x_k\} - \{x_i, x_j\} x_k \\
&\quad + (-1)^{|x_j||x_k|+|x_i||x_k|} x_k ((-1)^{|x_i||x_j|} x_j y_i + \{x_i, x_j\}) + (-1)^{|x_j||x_k|} \{x_i, x_k\} x_j \\
&\equiv 0 \mod(J; y_i y_j x_k).
\end{aligned}$$

□

Proposition 3.13. Let $R = T(V)/\langle x_\alpha \otimes x_\beta - (-1)^{|x_\alpha||x_\beta|} x_\beta \otimes x_\alpha \mid \forall \alpha, \beta \in \Lambda \rangle$ be a DG Poisson algebra with an arbitrary differential d and Poisson structure $\{\cdot, \cdot\}$. Suppose that $A = R/I$ is a DG Poisson homomorphic image of R , where I is a subset of monic elements of R . If $I \cup \psi(I)$ is closed under composition in R^e , then

$$\{u \in \mathfrak{B} \mid u \text{ is } I \cup \psi(I)\text{-standard}\}$$

forms a \mathbb{k} -linear basis of A^e , where \mathfrak{B} is the one given in Theorem 3.12.

Proof. It follows immediately from Proposition 3.4 and Theorem 3.12. □

Remark 3.14. Although the degree of the Poisson bracket is zero in our definition of DG Poisson algebra, the result obtained in this paper are also true for DG Poisson algebras of degree n with some expected signs, where $n \in \mathbb{Z}$ is the degree of the Poisson bracket.

Example 3.15. Let

$$A = \frac{\mathbb{k} \langle x_1, x_2 \rangle}{(x_1 x_2, x_2 x_1, x_2^2)},$$

where $|x_1| = 2, |x_2| = 3$. Let $d : A \rightarrow A$ be a \mathbb{k} -linear map of degree 1 by

$$d(x_1) = x_2, \quad d(x_2) = 0.$$

Moreover, we can define the Poisson bracket by

$$\{x_1, x_2\} = -\{x_2, x_1\} = x_2^2, \{x_1, x_1\} = \{x_2, x_2\} = 0.$$

Note that $0 = x_2^2 \in A$, it is easy to see that A is a DG Poisson algebra of degree 1, where the degree of the Poisson bracket is 1. Here, we can suppose that

$$R = \mathbb{k} \langle x_1, x_2 \rangle / (x_1 x_2 - x_2 x_1, x_2^2)$$

and $I = \langle x_1 x_2 \rangle$ is a DG Poisson ideal of R . Given an ordering on the set of generators of R^e by

$$x_1 < x_2 < y_1 < y_2.$$

Since the set consisting of $x_1 x_2, \psi(x_1 x_2) = x_1 y_2 + x_2 y_1$ is closed under composition in R^e , the universal enveloping algebra A^e has a \mathbb{k} -linear basis

$$\{x_1^{i_1} x_2^{i_2} y_1^{j_1} y_2^{j_2} \mid i_1 i_2 = 0, i_2 j_1 = 0\}$$

by Proposition 3.13.

Corollary 3.16. Let $R = T(V) / \langle x_\alpha \otimes x_\beta - (-1)^{|x_\alpha||x_\beta|} x_\beta \otimes x_\alpha \mid \forall \alpha, \beta \in \Lambda \rangle$ be a DG Poisson algebra with an arbitrary differential d and Poisson structure $\{\cdot, \cdot\}$. Then the universal enveloping algebra R^e is a DG free left and right R -module with basis

$$\mathfrak{C} = \{y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n} \mid j_r = 0, 1, 2, \dots\}.$$

Proof. By Theorem 3.12, R^e is a DG free left R -module. To show that R^e is a DG free right R -module, it is enough to prove that

$$\mathfrak{B}' = \{y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \mid j_r, i_r = 0, 1, 2, \dots\}$$

forms a \mathbb{k} -linear basis for R^e .

We show that \mathfrak{B}' is \mathbb{k} -linearly independent. Let

$$(3.1) \quad \sum_{1 \leq i_1 + \dots + i_n} \alpha_{i_1 \dots i_n} y_1^{i_1} \cdots y_n^{i_n} + \sum_{1 \leq j_1 + \dots + j_n, 1 \leq l_1 + \dots + l_n} \beta_{j_1 \dots j_n l_1 \dots l_n} y_1^{j_1} \cdots y_n^{j_n} x_1^{l_1} \cdots x_n^{l_n} \\ + \sum_{1 \leq m_1 + \dots + m_n} \gamma_{m_1 \dots m_n} x_1^{m_1} \cdots x_n^{m_n} + \delta 1 = 0,$$

where $\alpha_{i_1 \dots i_n}, \beta_{j_1 \dots j_n l_1 \dots l_n}, \gamma_{m_1 \dots m_n}$ and $\delta \in \mathbb{k}$. If there exist nonzero $\beta_{j_1 \dots j_n l_1 \dots l_n}$'s in the second term of formula (3.1), assume that

$$z = y_1^{j_1} \cdots y_n^{j_n} x_1^{l_1} \cdots x_n^{l_n}$$

is maximal among the monomials with nonzero coefficients in the second term of formula (3.1). Using (a) of Lemma 3.11, all monomials in the second term of formula (3.1) can be expressed as \mathbb{k} -linear combinations of monomials in \mathfrak{B} of Theorem 3.12. Then the coefficient of z is the coefficient of $x_1^{l_1} \cdots x_n^{l_n} y_1^{j_1} \cdots y_n^{j_n}$. Thus all $\beta_{j_1 \dots j_n l_1 \dots l_n}$'s in the second term of formula (3.1) are zero and hence all $\alpha_{i_1 \dots i_n}, \gamma_{m_1 \dots m_n}$ and δ in formula (3.1) are also zero by Theorem 3.12. By Lemma 3.11, it is easy to see that all monomials in \mathfrak{B} of Theorem 3.12 are spanned by \mathfrak{B}' . Therefore \mathfrak{B}' forms a \mathbb{k} -linear basis of R^e , as required. \square

4. SIMPLE DG POISSON MODULES

In this section, for a given DG Poisson algebra $A = R/I$, where R and A are defined as in Section 3, we prove that a DG symplectic ideal P of A is the annihilator of a simple DG Poisson A -module.

Lemma 4.1. Let \mathcal{A} be the set of all DG ideals in a DG Poisson algebra R and let \mathcal{B} be the set of all left DG ideals of R^e . For $I \in \mathcal{A}$, let \bar{I} be the left DG ideal of R^e generated by I . Then the map $I \rightarrow \bar{I}$ is an injective map from \mathcal{A} into \mathcal{B} and preserves inclusion, that is, $\bar{I} \cap R = I$. In particular, I is a DG Poisson ideal of R if and only if \bar{I} is an DG ideal of R^e .

Proof. Let $I \in \mathcal{A}$. Since R^e is a DG free right R -module with basis \mathfrak{C} given in Corollary 3.16, every element of $\bar{I} = R^e I$ is of the form

$$1a_0 + Y_1a_1 + Y_2a_2 + \cdots + Y_s a_s$$

for some $a_i \in I, 1 \neq Y_i \in \mathfrak{C}$. Hence, if $I \subseteq J$ then $\bar{I} \subseteq \bar{J}$ and $I = \bar{I} \cap R$ by Corollary 3.16, thus the map $I \rightarrow \bar{I}$ is injective. The second statement follows immediately from Lemma 3.11. \square

Remark 4.2. The map $I \rightarrow \bar{I}$ from \mathcal{A} into \mathcal{B} is not surjective. In [16], we can see that in the situation of the DG Poisson algebra concentrated in degree 0 with trivial differential, this map is not a surjective map.

Lemma 4.3. *Let (R^e, m_R, h_R) be the universal enveloping algebra of a DG Poisson algebra $(R, d, \{\cdot, \cdot\})$. If I is a DG Poisson ideal of R and Q is the DG ideal of R^e generated by I and $h_R(I)$, then $(R^e/Q, m', h')$ is the universal enveloping algebra of $A = R/I$, where*

$$\begin{aligned} m' : A &\rightarrow R^e/Q, \quad m'(r + I) = m_R(r) + Q, \\ h' : A &\rightarrow R^e/Q, \quad h'(r + I) = h_R(r) + Q. \end{aligned}$$

Proof. Since Q is the DG ideal of R^e generated by I and $h_R(I)$, we have

$$Q = R^e I + R^e h_R(I) R^e$$

by Lemma 4.1. Let $(D, \delta) \in \mathbf{DGA}$ with a DG algebra map $f : (A, d) \rightarrow (D, \delta)$ and a DG Lie algebra map $g : (A, \{\cdot, \cdot\}_A, d) \rightarrow (D_P, [\cdot, \cdot], \delta)$ satisfying

$$\begin{aligned} f(\{a, b\}) &= g(a)f(b) - (-1)^{|a||b|}f(b)g(a), \\ g(ab) &= f(a)g(b) + (-1)^{|a||b|}f(b)g(a), \end{aligned}$$

for any homogeneous elements $a, b \in A$. Denote by π' the canonical projection from R onto A , then there exists a unique DG algebra map ϕ from R^e into D such that $\phi m_R = f\pi'$ and $\phi h_R = g\pi'$ since (R^e, m_R, h_R) is the universal enveloping algebra of a DG Poisson algebra $(R, \{\cdot, \cdot\}, d)$. Hence

$$\phi(I) = \phi m_R(I) = f\pi'(I) = 0 \quad \text{and} \quad \phi h_R(I) = g\pi'(I) = 0,$$

so the DG ideal Q is contained in the kernel of ϕ . Thus ϕ induces the DG algebra map ϕ' from R^e/Q into D such that $\phi'(u + Q) = \phi(u)$ for all $u \in R^e$. Clearly, for all $r + I \in A$, we have that

$$\begin{aligned} \phi' m'(r + I) &= \phi'(m_R(r) + Q) = \phi(m_R(r)) = f\pi'(r) = f(r + I), \\ \phi' h'(r + I) &= \phi'(h_R(r) + Q) = \phi(h_R(r)) = g\pi'(r) = g(r + I). \end{aligned}$$

If ϕ_1 is an DG algebra map from R^e/Q into D such that $\phi_1 m' = f$ and $\phi_1 h' = g$, then we have $\phi_1 = \phi'$ since R^e/Q is generated by $m'(A)$ and $h'(A)$. It completes the proof. \square

Remark 4.4. From the Lemma 4.3, we have the following two observations:

- Since R^e is the universal enveloping algebra of a DG Poisson algebra R , there is an DG algebra homomorphism η from R^e into A^e such that $\eta m_R = m_A \pi'$ and $\eta h_R = h_A \pi'$.

$$\begin{array}{ccc} R & \xrightarrow{\pi'} & A \\ m_R, h_R \downarrow & & \downarrow m_A, h_A \\ R^e & \xrightarrow{\eta} & A^e \end{array}$$

- η is an epimorphism, since A^e is generated by $A = m_A(A)$ and $h_A(A)$. Set

$$I = \ker \pi', \quad Q = R^e I + R^e h_R(I) R^e.$$

We have $\ker(\eta) = Q$.

Lemma 4.5. *If f and g are DG \mathbb{k} -linear maps from a DG Poisson algebra A into a DG algebra B such that*

$$\begin{aligned} f(\{a, b\}) &= g(a)f(b) - (-1)^{|a||b|}f(b)g(a), \\ g(ab) &= f(a)g(b) + (-1)^{|a||b|}f(b)g(a), \end{aligned}$$

for any homogeneous elements $a, b \in A$, then

$$\begin{aligned} f(\{a, b\}) &= f(a)g(b) - (-1)^{|a||b|}g(b)f(a), \\ g(ab) &= g(a)f(b) + (-1)^{|a||b|}g(b)f(a). \end{aligned}$$

Proof. For any homogeneous elements $a, b \in A$, we have

$$\begin{aligned} f(\{a, b\}) + g(ab) &= g(a)f(b) + f(a)g(b), \\ f(\{b, a\}) + g(ba) &= g(b)f(a) + f(b)g(a). \end{aligned}$$

Note that

$$\{a, b\} = -(-1)^{|a||b|}\{b, a\}, \quad ab = (-1)^{|a||b|}ba.$$

Hence

$$\begin{aligned} 2(-1)^{|a||b|}f(\{a, b\}) &= (-1)^{|a||b|}g(a)f(b) + (-1)^{|a||b|}f(a)g(b) - g(b)f(a) - f(b)g(a) \\ &= (-1)^{|a||b|}f(a)g(b) - g(b)f(a) + (-1)^{|a||b|}(-1)^{|a||b|}f(\{a, b\}) \end{aligned}$$

and

$$\begin{aligned} 2(-1)^{|a||b|}g(ab) &= (-1)^{|a||b|}g(a)f(b) + (-1)^{|a||b|}f(a)g(b) + g(b)f(a) + f(b)g(a) \\ &= (-1)^{|a||b|}g(a)f(b) + g(b)f(a) + (-1)^{|a||b|}g(ab). \end{aligned}$$

Therefore, we have the conclusion. \square

Lemma 4.6. *Let (R^e, m_R, h_R) be the universal enveloping algebra of a DG Poisson algebra R . Then*

$$(4.1) \quad h_R(x_i^n) = \begin{cases} (k + k(-1)^{(2k-1)|x_i|^2})y_i x_i^{2k-1}, & \text{if } n = 2k, \\ (k + 1 + k(-1)^{(2k-1)|x_i|^2})y_i x_i^{2k}, & \text{if } n = 2k + 1. \end{cases}$$

Proof. Since

$$h_R(ab) = h_R(a)m_R(b) + (-1)^{|a||b|}h_R(b)m_R(a) = h_R(a)b + (-1)^{|a||b|}h_R(b)a$$

for all elements $a, b \in R$ by Lemma 4.5. Observe that formula (4.1) is true on $n = 1$ and $n = 2$ since

$$\begin{cases} h_R(x_i) = y_i, \\ h_R(x_i^2) = h_R(x_i)x_i + (-1)^{|x_i||x_i|}h_R(x_i)x_i = (1 + (-1)^{|x_i|^2})y_i x_i. \end{cases}$$

Set

$$\Delta(n) = \begin{cases} k + k(-1)^{(2k-1)|x_i|^2}, & \text{if } n = 2k, \\ k + 1 + k(-1)^{(2k-1)|x_i|^2}, & \text{if } n = 2k + 1. \end{cases}$$

Thus in order to complete the proof, we only need to show

$$h_R(x_i^{n+1}) = \Delta(n+1)y_i x_i^n$$

provided that $h_R(x_i^n) = \Delta(n)y_i x_i^{n-1}$.

Indeed if $n = 2k$, then

$$\begin{aligned} h_R(x_i^{2k+1}) &= h_R(x_i^{2k})x_i + (-1)^{|x_i^{2k}||x_i|} h_R(x_i)x_i^{2k} \\ &= \Delta(2k)y_i x_i^{2k-1} x_i + (-1)^{|x_i^{2k}||x_i|} y_i x_i^{2k} \\ &= (k+1+k(-1)^{(2k-1)|x_i|^2}) y_i x_i^{2k} \\ &= \Delta(n+1)y_i x_i^n. \end{aligned}$$

On the other hand, if $n = 2k+1$, then

$$\begin{aligned} h_R(x_i^{2k+2}) &= h_R(x_i^{2k+1})x_i + (-1)^{|x_i^{2k+1}||x_i|} h_R(x_i)x_i^{2k+1} \\ &= \Delta(2k+1)y_i x_i^{2k} x_i + (-1)^{|x_i^{2k+1}||x_i|} y_i x_i^{2k+1} \\ &= (k+1+(k+1)(-1)^{(2k+1)|x_i|^2}) y_i x_i^{2k+1} \\ &= \Delta(n+1)y_i x_i^n, \end{aligned}$$

as required. \square

Using the above notation, we have the following lemma.

Lemma 4.7. *Let (R^e, m_R, h_R) be the universal enveloping algebra of a DG Poisson algebra R . Then*

$$(4.2) \quad h_R(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}) = \sum_{k=1}^n (-1)^{|x_1^{i_1} \cdots x_{k-1}^{i_{k-1}}||x_k|} \Delta(i_k) y_k x_1^{i_1} \cdots x_{k-1}^{i_{k-1}} x_k^{i_k-1} x_{k+1}^{i_{k+1}} \cdots x_n^{i_n}.$$

Proof. For each monomial $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ of R , we proceed the proof using induction on the number of indeterminates. By Lemma 4.6, it is clear that formula (4.2) is true on the $x_1^{i_1}$. Thus in order to complete the proof, we only need to show the formula (4.2) on the $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ provided that

$$h_R(x_1^{i_1} x_2^{i_2} \cdots x_{n-1}^{i_{n-1}}) = \sum_{k=1}^{n-1} (-1)^{|x_1^{i_1} \cdots x_{k-1}^{i_{k-1}}||x_k|} \Delta(i_k) y_k x_1^{i_1} \cdots x_{k-1}^{i_{k-1}} x_k^{i_k-1} x_{k+1}^{i_{k+1}} \cdots x_{n-1}^{i_{n-1}}.$$

Since

$$h_R(ab) = h_R(a)m_R(b) + (-1)^{|a||b|} h_R(b)m_R(a) = h_R(a)b + (-1)^{|a||b|} h_R(b)a$$

for all elements $a, b \in R$ by Lemma 4.5, we have

$$\begin{aligned} h_R(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}) &= h_R(x_1^{i_1} x_2^{i_2} \cdots x_{n-1}^{i_{n-1}}) x_n^{i_n} + (-1)^{|x_1^{i_1} x_2^{i_2} \cdots x_{n-1}^{i_{n-1}}||x_n|} h_R(x_n^{i_n}) x_1^{i_1} x_2^{i_2} \cdots x_{n-1}^{i_{n-1}} \\ &= \left(\sum_{k=1}^{n-1} (-1)^{|x_1^{i_1} \cdots x_{k-1}^{i_{k-1}}||x_k|} \Delta(i_k) y_k x_1^{i_1} \cdots x_{k-1}^{i_{k-1}} x_k^{i_k-1} x_{k+1}^{i_{k+1}} \cdots x_{n-1}^{i_{n-1}} \right) x_n^{i_n} \\ &\quad + (-1)^{|x_1^{i_1} x_2^{i_2} \cdots x_{n-1}^{i_{n-1}}||x_n|} \Delta(i_n) y_n x_n^{i_n-1} x_1^{i_1} x_2^{i_2} \cdots x_{n-1}^{i_{n-1}} \\ &= \sum_{k=1}^{n-1} (-1)^{|x_1^{i_1} \cdots x_{k-1}^{i_{k-1}}||x_k|} \Delta(i_k) y_k x_1^{i_1} \cdots x_{k-1}^{i_{k-1}} x_k^{i_k-1} x_{k+1}^{i_{k+1}} \cdots x_{n-1}^{i_{n-1}} x_n^{i_n} \\ &\quad + (-1)^{|x_1^{i_1} x_2^{i_2} \cdots x_{n-1}^{i_{n-1}}||x_n|} \Delta(i_n) y_n x_1^{i_1} x_2^{i_2} \cdots x_{n-1}^{i_{n-1}} x_n^{i_n-1} \\ &= \sum_{k=1}^n (-1)^{|x_1^{i_1} \cdots x_{k-1}^{i_{k-1}}||x_k|} \Delta(i_k) y_k x_1^{i_1} \cdots x_{k-1}^{i_{k-1}} x_k^{i_k-1} x_{k+1}^{i_{k+1}} \cdots x_{n-1}^{i_{n-1}} x_n^{i_n}, \end{aligned}$$

as required. \square

Proposition 4.8. Let (R^e, m_R, h_R) be the universal enveloping algebra of a DG Poisson algebra R . Then, for every monomial $x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \in R$, the expression of $z = y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n} h_R(x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n})$ as a right R -combination of monomial in \mathfrak{C} of Corollary 3.16 is of the form

$$k_1 Y_1 a_1 + k_2 Y_2 a_2 + \cdots + k_s Y_s a_s, \quad k_i \in \mathbb{K}, \quad 1 \neq Y_i \in \mathfrak{C} \text{ and } a_i \in R.$$

That is, the coefficient of 1 under the expression of z as a right R -combination of monomials in \mathfrak{C} is zero.

Proof. Since

$$h_R(ab) = m_R(a)h_R(b) + (-1)^{|a||b|}m_R(b)h_R(a) = ah_R(b) + (-1)^{|a||b|}bh_R(a),$$

for all elements $a, b \in R$, we have $h_R(1) = 0$, that is, $h_R(k) = 0$ for every $k \in \mathbb{K}$. Express $h_R(x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n})$ as given in Lemma 4.7 and then it suffices to prove the statement for the case $z = y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n} y_i$.

For a monomial $y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n}$, we proceed the proof using induction on the $r = j_1 + j_2 + \cdots + j_n$. If $r = 0$ or 1, then it is trivial. Assume that the statement is true for monomials with length less than $j_1 + j_2 + \cdots + j_n$, we should proof the statement is also true for monomials with length equal $j_1 + j_2 + \cdots + j_n$. There are two possibilities: (I) $j_n \neq 0$, (II) $j_n = 0$ (without loss of generality, we can further assume that $j_{n-1} \neq 0$).

Case I. $j_n \neq 0$: Note that

$$y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n} y_i = y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n-1} ((-1)^{|x_n||x_i|} y_i y_n + h_R(\{x_n, x_i\}))$$

by Lemma 3.11. By the induction hypothesis, we have the expression of $y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n-1} y_i$ as a right R -combination of monomial in \mathfrak{C} of Corollary 3.16 is of the form

$$k_1 Y_1 a_1 + k_2 Y_2 a_2 + \cdots + k_s Y_s a_s, \quad k_i \in \mathbb{K}, \quad 1 \neq Y_i \in \mathfrak{C} \text{ and } a_i \in R.$$

Then the result on the case

$$(-1)^{|x_n||x_i|} y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n-1} y_i y_n$$

is true by (a) of Lemma 3.11. On the other hand, since $\{x_n, x_i\}$ can expressed as polynomial in R , we can see that the result on the case

$$y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n-1} h_R(\{x_n, x_i\})$$

is also true by Lemma 4.7 and the induction hypothesis.

Case II. $j_n = 0$: Note that

$$y_1^{j_1} y_2^{j_2} \cdots y_{n-1}^{j_{n-1}} y_i = y_1^{j_1} y_2^{j_2} \cdots y_{n-1}^{j_{n-1}-1} ((-1)^{|x_{n-1}||x_i|} y_i y_{n-1} + h_R(\{x_{n-1}, x_i\})).$$

Similarly, we have

$$\begin{aligned} y_1^{j_1} y_2^{j_2} \cdots y_{n-1}^{j_{n-1}-1} y_i y_{n-1} &= (k'_1 Y'_1 a'_1 + k'_2 Y'_2 a'_2 + \cdots + k'_s Y'_s a'_s) y_{n-1} \\ &= \sum_{i=1}^s (-1)^{|x_{n-1}||a'_i|} k'_i Y'_i (y_{n-1} a'_i - \{x_{n-1}, a'_i\}) \end{aligned}$$

by Lemma 3.11 and the induction hypothesis, where $k'_i \in \mathbb{K}$, $1 \neq Y'_i \in \mathfrak{C}$ and $a'_i \in R$. Thus, the result on the case

$$\sum_i (-1)^{|x_{n-1}||a'_i|} k'_i Y'_i y_{n-1} a'_i$$

is true by the case I. Further, similar to the proof of the case I, it is easy to see that the result on the case II is also true.

Therefore, we finish the proof. \square

Lemma 4.9. *Let (R^e, m_R, h_R) be the universal enveloping algebra of a DG Poisson algebra R . If $M \neq R$ is a DG ideal of R and Q is a DG Poisson ideal contained in M , then*

$$R^e M + R^e h_R(Q) R^e \neq R^e.$$

Proof. Since R^e is a DG free right R -module by Corollary 3.16, every element of $R^e M + R^e h_R(Q) R^e$ is of the form

$$\sum_i b_i m_i + c_i h_R(q_i) a_i, \quad a_i, b_i, c_i \in R^e, \quad m_i \in M, \quad q_i \in Q.$$

Express each a_i as a right R -combination of monomials Y_j in \mathfrak{C} of Corollary 3.16. Since

$$y_i h_R(q_i) = (-1)^{|x_i||q_i|} h_R(q_i) y_i + h_R(\{x_i, q_i\})$$

for each $i = 1, 2, \dots, n$ by (c) of Lemma 3.11 and since $\{x_i, q_i\} \in Q$, we may assume that $a_i \in R$ for all i . Next express each c_i as a right R -combination of monomials Y_j in \mathfrak{C} of Corollary 3.16. For $r \in R$, since

$$h_R(q_i) r = (-1)^{|r||q_i|} r h_R(q_i) + \{q_i, r\}$$

by (b) of Lemma 3.11 and $\{q_i, r\} \in Q \subseteq M$, we may assume that each c_i is a \mathbb{k} -linear combination of monomials in \mathfrak{C} of Corollary 3.16. Thus by Proposition 4.8, $\sum_i c_i h_R(q_i) a_i$ can be written as

$$\sum_i c_i h_R(q_i) a_i = \sum_i k'_i Y'_i r'_i, \quad k'_i \in \mathbb{k}, \quad 1 \neq Y'_i \in \mathfrak{C} \text{ and } r'_i \in R.$$

Finally, express each b_i as a right R -combination of monomials Y_j in \mathfrak{C} of Corollary 3.16. Then every element of $R^e M + R^e h_R(Q) R^e$ is of the form

$$1m_0 + k_1 Y_1 r_1 + k_2 Y_2 r_2 + \dots + k_s Y_s r_s, \quad m_0 \in M, \quad k_i \in \mathbb{k}, \quad 1 \neq Y_i \in \mathfrak{C} \text{ and } r_i \in R.$$

Thus $R^e M + R^e h_R(Q) R^e$ does not contain the unity since $M \neq R$ is a DG ideal of R . Therefore, we complete the proof. \square

Now, we study the simple DG Poisson module.

Let $(B, \cdot, \{\cdot, \cdot\}, d) \in \mathbf{DGPA}$ and let (B^e, m_B, h_B) be the universal enveloping algebra of B , the definition of a DG Poisson module M over a DG Poisson algebra B is given in Definition 2.4. Note that the annihilator of a DG Poisson B -module M is defined to be

$$\text{ann}_B(M) = \{b \in B \mid b \cdot M = 0\},$$

which is a DG Poisson ideal of B . Since M is a left DG B^e -module, it is easy to prove that

$$\text{ann}_B(M) = m_B^{-1}(\text{ann}_{B^e}(M)) = \text{ann}_{B^e}(M) \cap B,$$

where B is a finitely generated DG Poisson algebra.

Now follow the idea of [16], we can define a DG symplectic ideal. A DG Poisson ideal Q of B is said to be DG symplectic ideal if there is a maximal DG ideal M of $(B, \cdot, \{\cdot, \cdot\}, d)$ such that Q is the largest DG Poisson ideal contained in M . Note that a simple DG Poisson module over a DG Poisson algebra B is the DG Poisson module over B that has no non-zero proper DG Poisson submodule.

Theorem 4.10. *Let $R = T(V)/\langle x_\alpha \otimes x_\beta - (-1)^{|x_\alpha||x_\beta|} x_\beta \otimes x_\alpha \mid \forall \alpha, \beta \in \Lambda \rangle$ be a DG Poisson algebra with an arbitrary differential d and Poisson structure $\{\cdot, \cdot\}$. Suppose that $A = R/I$ is a DG Poisson homomorphic image of R . If P is a DG symplectic ideal of A , then P is the annihilator of a simple DG Poisson A -module.*

Proof. From the above, let π' be the canonical projection from R onto A and let M be a maximal DG ideal of A such that P is the largest DG Poisson ideal contained in M . By Remark 4.4, there is an epimorphism η from R^e onto A^e such that $\ker(\eta) = R^e I + R^e h_R(I)R^e$.

$$\begin{array}{ccc} R & \xrightarrow{\pi'} & A \\ \downarrow m_R, h_R & & \downarrow m_A, h_A \\ R^e & \xrightarrow{\eta} & A^e \end{array}$$

Note that $\pi'^{-1}(M)$ is a maximal DG ideal of R and $\pi'^{-1}(P)$ is the largest DG Poisson ideal contained in $\pi'^{-1}(M)$. Since

$$R^e \pi'^{-1}(M) + R^e h_R(\pi'^{-1}(P))R^e \neq R^e$$

by Lemma 4.9, there is a maximal left DG ideal N of R^e containing $R^e \pi'^{-1}(M) + R^e h_R(\pi'^{-1}(P))R^e$. Then $X = R^e/N$ is a simple left DG R^e -module and so X is a simple left DG Poisson R -module.

Note that $\text{ann}_{R^e}(X)$ is the largest DG ideal of R^e contained in N :

since $\text{ann}_{R^e}(X) = \{u \in R^e \mid u \cdot X = 0\}$, we have

$$0 = u(u' + N) = uu' + N$$

for all $u' + N \in X$, then $uu' \in N$. Set $u' = 1$, we have $u \in N$, that means $\text{ann}_{R^e}(X) \subseteq N$. If there exist a DG ideal B strictly contained in N such that $\text{ann}_{R^e}(X) \subseteq B$, then there exist a $b \in B$ such that $b \notin \text{ann}_{R^e}(X)$, we have $b(u_1 + N) \neq 0$ for some $u_1 + N \in X$, thus $bu_1 \notin N$. But $b \in B, u_1 \in R^e, B$ is a DG ideal, then $bu_1 \in B \subseteq N$, which contradicts $bu_1 \notin N$. Hence $\text{ann}_{R^e}(X)$ is the largest DG ideal of R^e contained in N .

Since $R^e \pi'^{-1}(P)$ is a DG ideal by Lemma 4.1 and $R^e \pi'^{-1}(P) \subseteq R^e \pi'^{-1}(M) \subseteq N$, we have

$$\pi'^{-1}(P) = (R^e \pi'^{-1}(P)) \cap R \subseteq \text{ann}_{R^e}(X) \cap R \subseteq N \cap R$$

by Lemma 4.1. Since $\pi'^{-1}(M) \subseteq N \cap R \neq R$ and $\pi'^{-1}(M)$ is a maximal DG ideal of R , we have $\pi'^{-1}(M) = N \cap R$. Hence $\pi'^{-1}(P) = \text{ann}_{R^e}(X) \cap R = \text{ann}_R(X)$ since $\pi'^{-1}(P)$ is the largest DG Poisson ideal contained in $\pi'^{-1}(M)$.

Note that $\ker(\eta)$ is a DG ideal of R^e contained in N by Remark 4.4 since $I = \ker(\pi') \subseteq \pi'^{-1}(P)$ and $I \subseteq \pi'^{-1}(M)$. Hence $\ker(\eta)X = 0$ and so X is a simple left DG A^e -module with module structure induced by η . Thus X becomes a simple DG Poisson A -module.

Moreover, we have

$$\begin{aligned} \text{ann}_A(X) &= m_A^{-1}(\text{ann}_{A^e}(X)) = m_A^{-1}(\eta(\text{ann}_{R^e}(X))) \\ &= \pi' m_R^{-1}(\text{ann}_{R^e}(X)) = \pi'(\text{ann}_R(X)) = \pi' \pi'^{-1}(P) = P, \end{aligned}$$

which completes the proof. \square

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